ALEXANDER MEYER-GOHDE
DANIEL NEUHOFF

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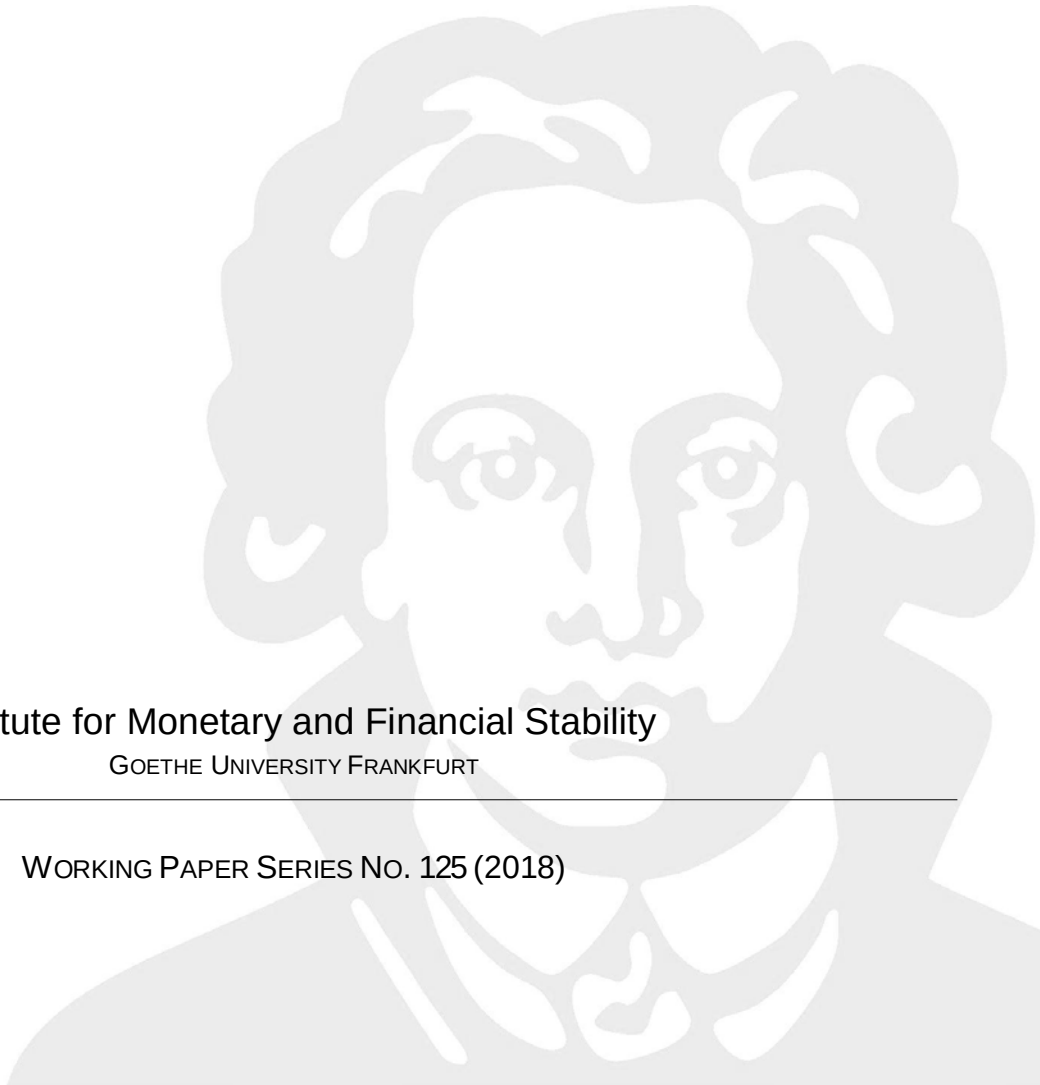

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Institute for Monetary and Financial Stability
Goethe University Frankfurt
House of Finance
Theodor-W.-Adorno-Platz 3
D-60629 Frankfurt am Main
www.imfs-frankfurt.de **|** info@imfs-frankfurt.de

Generalized Exogenous Processes in DSGE: A Bayesian Approach *

Alexander Meyer-Gohde[†§] Daniel Neuhoff[‡]

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Abstract

We relax the standard assumption in the dynamic stochastic general equilibrium (DSGE) literature that exogenous processes are governed by AR(1) processes and estimate ARMA (p,q) orders and parameters of exogenous processes. Methodologically, we contribute to the Bayesian DSGE literature by using Reversible Jump Markov Chain Monte Carlo (RJMCMC) to sample from the unknown ARMA orders and their associated parameter spaces of varying dimensions. In estimating the technology process in the neoclassical growth model using post war US GDP data, we cast considerable doubt on the standard AR(1) assumption in favor of higher order processes. We find that the posterior concentrates density on hump-shaped impulse responses for all endogenous variables, consistent with alternative empirical estimates and the rigidities behind many richer structural models. Sampling from noninvertible MA representations, a negative response of hours to a positive technology shock is contained within the posterior credible set. While the posterior contains significant uncertainty regarding the exact order, our results are insensitive to the choice of data filter; this contrasts with our ARMA estimates of GDP itself, which vary significantly depending on the choice of HP or first difference filter.

JEL classification: C11; C32; C51; C52

Keywords: Bayesian analysis; Dynamic stochastic general equilibrium model; Model evaluation; ARMA; Reversible Jump Markov Chain Monte Carlo

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[†]Goethe-Universität Frankfurt, Chair of Financial Markets and Macroeconomics, Theodor-W.-Adorno-Platz 3, 60629 Frankfurt am Main, Germany; Tel.: +49-69 798 34501; E-Mail: meyer-gohde@econ.uni-frankfurt.de

[§]Institute for Monetary and Financial Stability, Theodor-W.-Adorno-Platz 3, 60629 Frankfurt am Main, Germany

[‡]Humboldt-Universität zu Berlin, Institut für Wirtschaftstheorie II, Spandauer Straße 1, 10178 Berlin, Germany; Tel.: +49-30-2093 5667; E-Mail:daniel.neuhoff@gmail.com

1 Introduction

Despite recent advances in improving the fit of DSGE models to the data, misspecification remains. These structural models deliver structural predictions for the path of endogenous variables using a theory of agents' decision making processes, taking exogenous processes as given. As theory imposes no restrictions on these processes, their specification must be assumed by the modeler and standard practice is to impose an AR(1) form. In his Nobel Prize Lecture, Sims (2012, p. 1202) observes that "DSGEs could be made to fit better by adding parameters allowing more dynamics in the disturbances." Likewise, Del Negro and Schorfheide (2009) identify three approaches to deal with misspecification in rational expectations models: ignore it, generalize the stochastic driving forces, or relax the cross-equation restrictions. Most analyses take the first route, ignoring this issue altogether. While Del Negro and Schorfheide (2006) follow the third route with their DSGE-VAR, the DSGE literature has not yet provided a systematic framework to address the second approach to misspecification of generalizing stochastic driving forces.[1] In a closely related study, Cúrdia and Reis (2010) examine correlated exogenous processes and allow their vector of processes to follow vector AR processes out to order six, but still must impose a particular order prior to estimation and leave the question of estimating the order and the potential effects of moving average terms unanswered.[2] We fill this gap by estimating the order as well as the parameters of generalized ARMA representations of exogenous driving forces within DSGE models. Taking a Bayesian perspective, our posterior over the orders provides a quantification of model uncertainty.[3]

To accomplish the task, we adopt the Reversible Jump Markov Chain Monte Carlo (RJMCMC) methodology as pioneered by Green (1995).[4] RJMCMC provides samples from a posterior distribu-

[1] Some notable exceptions to the standard practice of assuming AR(1) or white noise processes for exogenous processes include Smets and Wouters (2007) who have the price-markup disturbance follow an ARMA(1,1) process, Del Negro and Schorfheide (2009) who let government expenditures follow an AR(2) instead of an AR(1) process, Justiniano, Primiceri, and Tambalotti (2008) who examine the robustness of the ARMA(1,1) specification for the wage and price markup shocks in the Smets and Wouters (2007) model, Croce's (2014) who models long-run growth as an ARMA(1,1) process.

[2] They do allow marginal likelihoods of the model to inform their choice of which estimated model to analyze and note that the highest order, six, is associated with the highest marginal likelihood. Nonetheless, they chose to analyze the first order process due to the ease of interpretation.

[3] While there are certainly alternatives to our Bayesian approach, for example selecting the model with the highest maximized likelihood or using model selection criteria like the Akaike Information Criterion, we incorporate model uncertainty into the inference of statistics of interest.

[4] Markov Chain Monte Carlo (MCMC) methods have become increasingly popular for the estimation of DSGE models in recent years. See especially Fernández-Villaverde and Rubio-Ramírez (2004); along with An and Schorfheide (2007), Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez (2010), Del Negro and Schorfheide (2011), and Guerrón-Quintana and Nason (2013) for methodological reviews and introductions; and Herbst and Schorfheide (2015)

tion spanning several, not necessarily nested, models with parameter spaces of potentially different dimensionality. In our case, each model is identified by a specific set of orders for the lag polynomials of the autoregressive and moving average components of the exogenous process, each leading to a different dimensionality of the parameter space—e.g., increasing the AR order introduces an additional parameter, increasing the dimensionality of the parameter space. The RJMCMC method rests on modifying the proposal ratios in the acceptance probability by inflating parameter vectors to common dimensionality in order to circumvent the dimensionality mismatch. We apply this approach to systematically explore the fit of DSGE models using different structures for the shock processes which, as emphasized by Ehlers and Brooks (2004), provides a computationally feasible alternative to estimating all different possible combinations of shock orders individually.[5] Again having a set of draws from the posterior allows us to quantify posterior model uncertainty and, additionally, its consequences for impulse responses and correlation structures.[6]

We begin by estimating ARMA representations of US post war GDP; we stationarize the data using two different filters, the first difference and HP filter. The posterior mode models are AR(2) and ARMA (4,5) for first differenced and HP filtered data respectively. We find that RJMCMC provides point estimates of the ARMA orders with a reliability comparable to traditional order selection criteria such as the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the Schwarz Criterion (SC). RJMCMC, in contrast, provides more than just point estimates by providing draws from the posterior distribution over different ARMA orders, and we find that the HP filtered GDP data is associated with substantial posterior model uncertainty with a more dispersed posterior.

We then turn to a prototypical DSGE model, Hansen's (1985) specification of the neoclassical growth model, and relax the traditional AR(1) assumption imposed on the exogenous technology

for a textbook treatment.

[5]The RJMCMC algorithm allows us to explore the posterior adaptively, which allows for a more efficient means of sampling across models than generating samples from the posterior of each model (for us, ARMA order combinations p and q) and then weighting according to Bayes factors— see Bauer (2016) for an empirical analysis of the term structure of interest rates using RJMCMC. Further computational efficiency gains could be achieved, for example, with Stephens's (2000) continuous birth-death algorithm for changes in the order of processes—see Vermaak, Andrieu, Doucet, and Godsill (2004) and Philippe (2006) for applications to autoregressive models.

[6]If multiple shocks are kept independent while generalizing their individual autocorrelation patterns, the resulting estimates admit a structural interpretation of the shocks that can guide the researcher in identifying those dimensions along which the model requires the most additional internal propagation. It may, furthermore, be possible to construct model selection criteria based on the comparison of the spectrum of variables of interest derived from estimates of the posterior with the spectrum using only pure white noise shocks giving a measure of how much structure has to be added to the model outside of economic theory, an idea along the lines of Watson (1993).

process. We choose this model as it has known misspecifications (e.g., a lack of persistence in output) and its simplicity enables us to focus on the application of the RJMCMC methodology. After confirming that RJMCMC would correctly identify the ARMA order using synthetic data generated from an AR(1) technology process, we turn to US post war GDP data and estimate the order and parameters of the technology process. We find that the data prefers higher order exogenous processes—at the mode, ARMA(3,0), but with substantial posterior density associated with other higher order specifications, such as ARMA(2,2).[7] Strikingly, these results are not sensitive to the choice of the HP versus first difference filter, with the same points estimates of order, associated parameters, and dispersion of the posterior for both. The higher order processes imply qualitatively different reactions of endogenous variables to technology shocks than under an AR(1) specification. Namely, the posterior impulse responses of all variables are hump shaped, reflecting common wisdom in the macroeconomics literature,[8] in contrast to the monotonic responses of, say, output or labor with the traditional AR(1) process.

From a DSGE likelihood perspective, there is, without a commensurate prior specification, no reason to prefer invertible or "fundamental" representations in the presence of MA terms. Accordingly, we sample from the covariance equivalent, noninvertible representations for draws with nonzero MA order. We find a downward shift in the amplitude of the impulse responses as well as an overall increase in the posterior uncertainty regarding the impulse responses of endogenous variables to a technology shock. Strikingly, we cannot exclude the possibility of a negative response of hours to a positive technology shock, with the noninvertible MA representations closest in spirit to the news shock hypothesis.[9] Thus, we fail to reject the Galí (1999) hypothesis of a decline in hours worked in response to a technology shock, even when the impulse responses are identified by the canonical RBC model.

Our approach can be considered a Bayesian Model Averaging (BMA) method[10] for providing impulse responses and moments under model uncertainty, in that we weigh these statistics from

[7] The internal propagation of the model through capital accumulation drives the tendency towards lower orders on the technology process than when estimating the orders of output directly as above.

[8] See especially, Cogley and Nason (1995).

[9] See, e.g., Beaudry and Portier (2005) or Barsky and Sims (2011).

[10] For an overview see Hoeting, Raftery, Madigan, and Volinsky (1999) who also document an improved out-of-sample forecasting performance using BMA, which is also found by Madigan and Raftery (1994) in the context of graphical models. Kass and Raftery (1995) provide a discussion of Bayesian model selection and averaging and Raftery, Madigan, and Hoeting (1997) discuss the merits of BMA in the context of linear regression models. Recent applications of RJMCMC in economics include instrumental variable regression by Koop, Leon-Gonzalez, and Strachan (2012) and variable selection in a dynamic, affine term structure model by Bauer (2016).

different models with their respective posterior probabilities. The BMA paradigm was put forth by Leamer (1978) and interest in this approach has since increased with the advent of more powerful MCMC samplers. In a DSGE context, Wolters (2015) uses BMA to provide meta forecasts using multiple estimated DSGE models and Strachan and Van Dijk (2013) use BMA with VARs to assess the empirical support for structural breaks and the long run and equilibria restrictions implied by a prototypical DSGE model. Our analysis is close in spirit to the latter; yet, whereas they apply BMA to estimate VARs restricted commensurate with a DSGE model or to provide forecasts using various estimated DSGE models, we apply BMA to estimate the DSGE model itself.

This paper is organized as follows: We first introduce our methodology and shortly illustrate the method by constructing a sampler for a univariate autoregressive model of unknown order. Afterwards, we present the results of a small Monte Carlo study designed to gauge the power of the method for identifying univariate autoregressive moving average models using synthetic data derived from estimated ARMA models of post war US GDP data. Lastly, we apply the method to the neoclassical growth model, using synthetic AR (1) as well as post war US data, and analyze the posterior model uncertainty and its consequences for posterior impulse responses and correlations.

2 Reversible Jump MCMC for ARMA Processes

The Reversible Jump Markov Chain Monte Carlo (RJMCMC) methodology pioneered by Green (1995) generalizes the Metropolis-Hastings algorithm (Hastings 1970) to allow for moves between parameter spaces of varying dimensionality while maintaining detailed balance.[11] This transdimensionality enables inference on a posterior distribution spanning several, not necessarily nested, models. In the following, we will illustrate the mechanics of RJMCMC with the construction of a sampler for univariate autoregressive models of unknown order using an RJMCMC approach.[12]

[11] A more extensive treatment of Metropolis-Hastings samplers can be found in Chib and Greenberg (1995). See also Tierney (1998) for a comparison of RJMCMC and conventional Metropolis-Hastings kernels. Another popular MCMC method is the Gibbs sampler which is a special case of Metropolis-Hastings samplers and ultimately RJMCMC samplers. See Gelfand and Smith (1990) for a review and comparison of Gibbs samplers as well as importance samplers and stochastic substitution and Troughton and Godsill (1998) for application to autoregressive models. Geweke (1998) provides an overview over Bayesian methods and their applications in economics. An and Schorfheide (2007) and Herbst and Schorfheide (2015), as well as Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez (2010), Del Negro and Schorfheide (2011), and Guerrón-Quintana and Nason (2013), provide extensive treatments and introductions in the context of DSGE modeling.

[12] Several authors have applied RJMCMC to the problem of estimating univariate autoregressive (moving average) models, e.g., Brooks, Giudici, and Roberts (2003), Ehlers and Brooks (2004), and Ehlers and Brooks (2008).Relatedly,

For illustration, we will derive our transdimensional random walk sampler implementation of the RJMCMC with a univariate zero mean normally distributed AR(p) model of unknown order p.[13] Such an AR(p) model is defined as

$$y_t = P_1^p y_{t-1} + P_2^p y_{t-2} + \ldots + P_p^p y_{t-p} + \epsilon_t, \quad \epsilon_t \sim \mathcal{N}\left(0, \sigma^2\right) \tag{1}$$

P_i^p are the coefficients of the lag polynomial of order p associated with the i'th lag and ϵ_t is a zero mean stochastic disturbance. Denote by $P^p \doteq \left\{P_1^p, P_2^p, \ldots, P_p^p\right\}$ the vector of parameters of the AR(p) model.[14] We would like to construct a posterior distribution over the orders, p, and associated parameters, P^p, given observations on y_t.

We interpret the order of the lag polynomial p as a model indicator and will use the terms model indicator and polynomial or lag order interchangeably. The aim is now to construct a sampler for the joint posterior distribution over the different models indexed by p and their parameters. The strategy closely resembles that for Metropolis-Hastings samplers.[15] Indeed, Metropolis-Hastings samplers are a special case in the RJMCMC framework. It is expositionally convenient to express the state of the Markov chain as

$$\varsigma = (p, P^p) \tag{2}$$

explicitly including the order of the autoregressive polynomial p in the state.

The detailed balance condition poses the main obstacle to the ability of transdimensional sampling to construct a joint posterior distribution over potentially nonnested models with parameter spaces of varying dimensionality. Recall the detailed balance condition (A-3),

$$\int_{\mathcal{A}} \pi(\varsigma) K(\varsigma, \mathcal{B}) d\varsigma = \int_{\mathcal{B}} \pi(\varsigma') K(\varsigma', \mathcal{A}) d\varsigma' \tag{3}$$

Unlike in the foregoing section, the dimension of ς can change. I.e., the state space of the Markov chain spans parameter spaces with differing dimensionality—for a sampler for AR(p) models of unknown order, when p changes so does the number of parameters. Here, the usual strategy for the derivation of the acceptance probability will fail. Green (1995) modifies the proposals in such a way that the integrals on both sides of the detailed balance condition are over spaces of the same

different approaches to statistical models of varying dimensionality have emerged; such as birth-death Markov Chain Monte Carlo, based on continuous time birth-death processes, as initiated by Stephens (2000) and applied to the analysis of autoregressive moving average models by Vermaak, Andrieu, Doucet, and Godsill (2004) and Philippe (2006). A summary and comparison of these methods can be found in Cappè, Robert, and Rydèn (2003).

[13] Our derivation follows the exposition of Waagepetersen and Sorensen (2001).

[14] The part of the parameter vector associated with the standard deviation of the disturbance ϵ_t, σ will be left implicit in the exposition of this section to maintain the focus on the order, p.

[15] The appendix contains a short description of conventional Metropolis-Hastings samplers to contrast with the RJMCMC algorithm provided here.

dimensionality by introducing an auxiliary proposal variable u together with a mapping $g_{pp'}$ that maps the auxiliary proposal u and the current state of the chain to the new proposed state. The mapping $g_{pp'}$ is chosen such that the dimensionality of the integrals on both sides of the equation is inflated to some common, potentially higher, dimensionality.

In order to be able to easily verify adherence to detailed balance for a move from a state (p, P^p) to $(p', P^{p'})$, the vectors of Markov chain states and the random auxiliary proposal variables (P^p, u) and $(P^{p'}, u')$ must be of equal dimension. This dimension matching condition ensures that $\pi(P^p|p)\gamma_{pp'}(P^p, u)$ and $\pi(P^{p'}|p')\gamma_{p'p}(P^{p'}, u')$ are "joint densities on spaces of equal dimension," (Waagepetersen and Sorensen 2001, p. 54) allowing an application of a change of variables in the detailed balance equation to facilitate the construction of the transition kernel of the Markov chain. Here, $\gamma_{pp'}(P^p, u)$ is the proposal density for the auxiliary variable u going from an AR model of order p to one with order p' which may also depend on the current parameter vector P^p. The proposed new order p' is drawn from some $\gamma_p(p'|p)$ and the joint proposal density is $\gamma(\varsigma) = \gamma_{pp'}(P^p, u)\gamma_p(p'|p)$.

In our implementation of the method, we use the following differentiable bijection for $g_{pp'}$

$$\begin{bmatrix} P^{p'} \\ u' \end{bmatrix} = g_{pp'}(P^p, u) = \begin{bmatrix} A(p, p')_{p'\times p} & I_{p'\times p'} \\ I_{p\times p} & 0_{p\times p'} \end{bmatrix} \begin{bmatrix} P^p \\ u \end{bmatrix} \tag{4}$$

where

$$A(p, p') = \begin{cases} \begin{bmatrix} I_{p\times p} \\ 0_{(p'-p)\times p} \end{bmatrix} & \text{if } p' > p \\ \begin{bmatrix} I_{p'\times p'} 0_{p'\times(p-p')} \end{bmatrix} & \text{if } p' < p \\ \begin{bmatrix} I_{p'\times p'} \end{bmatrix} & \text{if } p' = p \end{cases} \tag{5}$$

This mapping leads to the transdimensional analog of a full-site updating random walk sampler. Proposals for "newly born" parameters, i.e., those $P_i^{p'}$ for $i = p+1, \ldots, p'$, are centered around zero. If $p' < p$ the parameter vector is truncated and proposals for these parameters are centered around their previous values. For $p' = p$ this mapping gives a standard random walk sampler.

The detailed balance condition holds if[16]

$$\int_{\mathcal{A}_p} \pi(\varsigma)\, Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int_{\mathcal{B}_{p'}} \pi(\varsigma')\, Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} \tag{6}$$

for all subsets $\mathcal{A}_p$ and $\mathcal{B}_{p'}$ of the parameter spaces associated with autoregressive polynomials of order p and p' respectively and where

$$Q\left(\varsigma, \mathcal{B}_{p'}\right) = \int_{\mathcal{B}_{p'}} \gamma(\varsigma'|p, P^p)\alpha_{pp'}(\varsigma, \varsigma')d\varsigma'$$

is the first part of the kernel in (A-5); i.e., the part of the conditional distribution of ς' associated

[16] See also Waagepetersen and Sorensen (2001).

with acceptance of the proposal.

Implementing the change of variables with the mapping $g_{pp'}$, the detailed balance condition is satisfied if

$$\pi(\varsigma)\,\gamma_p(p'|p)\alpha_{pp'}\gamma_{pp'}(P^p,u) = \pi(\varsigma')\,\gamma_p(p|p')\alpha_{p'p}\gamma_{p'p}(g_{pp'}(P^p,u)) \tag{7}$$

where the details of the derivation can be found in the appendix.

Following Peskun (1973), we set the acceptance probability $\alpha_{pp'}$ as large as possible,[17]

$$\alpha_{pp'} = \min\left(1, \chi_{pp'}(\varsigma,\varsigma')\right) \tag{8}$$

with

$$\chi_{pp'}(\varsigma,\varsigma') = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \times \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \times \underbrace{\frac{\gamma_p(p|p')\gamma_{p'p}(g_{pp'}(P^p,u))}{\gamma_p(p'|p)\gamma_{pp'}(P^p,u)}}_{\text{Proposal Ratio}} \tag{9}$$

Having chosen an appropriate acceptance probability to maintain detailed balanced, we can now implement the procedure. The resulting sequence of states approximates the joint posterior over all models indexed by their order p and the corresponding parameter vectors.

RJMCMC Algorithm

1. Set the initial state ς_0 of the Markov chain
2. For $i = 1$ to N
   (a) set $\varsigma = \varsigma_{i-1}$
   (b) Propose a visit to model p' with probability $\gamma_p(p'|p)$
   (c) Sample u from $\gamma_{pp'}(P^p,u)$
   (d) Set $(P',u') = g_{pp'}(P^p,u)$
   (e) Accept draw with probability
   $$\alpha = \min\left(1, \chi_{pp'}(\varsigma,\varsigma')\right)$$
   $\chi_{pp'}$ is defined as in (9)
   (f) If the draw is accepted set $\varsigma_i = \varsigma'$. If the draw is rejected set $\varsigma_i = \varsigma$

The application to moving average models follows by analogy and the extension to autoregressive moving average (ARMA) models is straightforward. One simply defines the model indicator as a two-element vector, proposing not only visits to some model with autoregressive order p' but also for a new order for the MA-polynomial q'. Likewise, moving from scalar to vector ARMA processes of unknown order could entail choosing the maximum AR and MA orders over all the processes, in which the model indicator would remain a two-element vector and the paramters of the model would become matrices, or each process could have an individual model indication, making this now a two-n-element vector for n individual processes.

[17] Which, as noted by Green (1995), is "optimal in the sense of reducing the autocorrelation of the chain."

For many applications, it is desirable to restrict the parameter spaces of ARMA processes to ensure stationarity and/or invertibility.[18] To constrain sampling to these invertible and stationary regions of the parameters spaces of each model, we reparametrize the AR (and MA) polynomials in terms of their (inverse) partial autocorrelations (PACs). Details, again, are in the appendix.

3 RJMCMC ARMA Order and Parameter Estimation: Monte Carlo Evidence

We examine the performance of the RJMCMC method for ARMA processes of unknown order introduced in the foregoing section by carrying out two Monte Carlo experiments. For both experiments, we compare the model chosen by the posterior mode of our RJMCMC algorithm with the choices that follow from using the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the Schwarz Criterion (SC). We orient the Monte Carlo experiments around the same post war US per capita real GDP data[19] that will inform our DSGE model in the following section by applying our RJMCMC algorithm to obtain draws from the posterior distribution of demeaned first-differenced and HP filtered quarterly observations of US post war per capita real GDP. We find that the RJMCMC algorithm performs favorably in comparison with the three standard alternatives for identifying the ARMA orders.

3.1 Priors and Proposals

Table 1 summarizes the priors and proposals that we used in the Monte Carlo study. We choose a uniform prior over the AR and MA orders, restricting the highest allowed order to 10 for both the AR and MA polynomials. Proposals for the AR and MA orders are taken to follow a discretized

[18] For the DSGE application in sections 4 and 5, we will require stationarity of the exogenous driving forces. In section 5, we will examine the consequences of imposing or not imposing invertibility on MA components, should they exist, on impulse responses.

[19] We take 1947:1-2013:3 real GDP from the NIPA tables, expressed on a per capita basis using the BLS series on the civilian noninstitutional population. Both data sets were downloaded from the St. Louis Federal Reserve's FRED database.

Variable	Prior	Proposal
p	U(0,10)	LaplaceD(p,2)
q	U(0,10)	LaplaceD(q,2)
AR PAC	TN(0,0.25)	TN(PAC,0.0025)
MA inverse PAC	TN(0,0.25)	TN(PAC,0.0025)
σ: Standard Deviation ϵ_t	IG(1,1)	TN(σ,0.0025)

Table 1: Prior and proposal distributions for ARMA GDP estimation

Laplace distribution, LaplaceD(μ, b),[20] with location parameter, μ, and shape parameter, b, such that

$$\gamma_p(p'|p) \propto exp(-b|p - p'|) \text{ with } p', p \in [0, 1, \ldots, 10] \tag{10}$$

$$\gamma_q(q'|q) \propto exp(-b|q - q'|) \text{ with } q', q \in [0, 1, \ldots, 10] \tag{11}$$

For the (inverse) partial autocorrelations, our prior is a truncated normal distribution, TN($\mu, \sigma, -1, 1$), with location parameter, μ, and dispersion σ, and truncations at 1 and -1, imposing invertibility and stationarity. With these proposal distributions, we center the (inverse) partial autocorrelations around their previous values and new (inverse) partial autocorrelations are centered around zero.

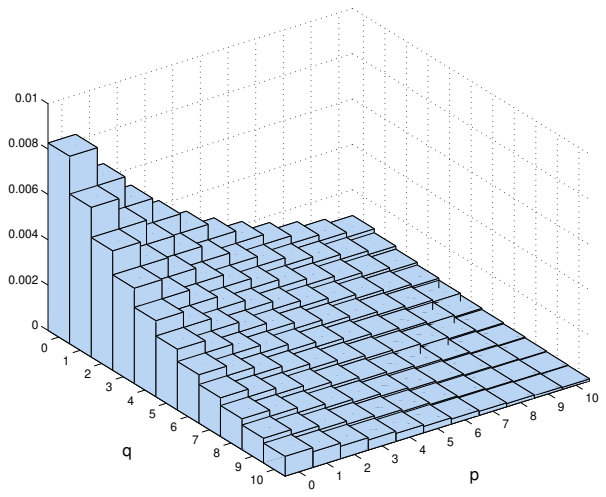

Figure 1: Implied prior over the orders p, q

All three standard information criteria penalize for the number of parameters in the model. This

[20] We choose the discretized Laplace over, say, a uniform over the (0,10) interval following Ehlers and Brooks (2002), who reiterate Troughton and Godsill's (1998) point that the dicretized Laplace places highest probability close to the current model, reducing the computational resources spent exploring models with a low posterior density. Consult Ehlers and Brooks (2002) for a study of different prior and proposal distributions in the context of autoregressive models.

feature is also present in the posterior of our RJMCMC method with proper priors over the (inverse) partial autocorrelations. Increasing the order of, say, an autoregressive model and setting the new parameter to zero gives a model identical to the previous one with lower order; hence, does not change the likelihood. Yet, the posterior with the additional parameter is penalized as the prior probability assigned to the value of the new parameter is smaller than one, yielding a posterior probability lower than with the original, lower order. Even though the prior on the orders is uniform, the prior over the parameters behaves implicitly like an exponential prior over the sum of the orders $(p + q)$ as shown in figure (1).

3.2 ARMA Posterior Mode Estimates of US GDP

We apply the RJMCMC algorithm to obtain 3,000,000 draws from the posterior distribution of first differenced and demeaned quarterly observations of the logarithm of US per capita real GDP as well as 7,000,000 draws from the posterior distribution of the cyclical component of US GDP extracted using an HP filter with the smoothing parameter set to 1600 for the period from 1947:1 - 2013:3.[21] We employ the Kalman filter to evaluate the log likelihood, stacking higher AR and MA lags to obtain a first order vector state space.

The posterior over models using first-differenced data can be found in figure (2). Note that there is a substantial amount of posterior uncertainty regarding the model with textbook representations such as Blanchard and Fischer's (1989, p. 9) ARMA(2,2) estimated on first differenced log GNP comfortably in the posterior distribution over models. The model at our posterior mode is an AR(2) with the posterior mean parameters conditional on this model being

$$y_t = 0.3184 y_{t-1} + 0.1297 y_{t-2} + \epsilon_t; \epsilon_t \sim N(0, 0.9025)$$

Figure (3) shows the posterior distribution over the orders p, q for the HP filtered data. Clearly, there is significant posterior uncertainty regarding the model reflected in the dispersion of poste-

[21] We generated more draws using the HP filtered data to compensate for its reduced acceptance rate. While the RJMCMC algorithm produced a total acceptance rate of 30% and an acceptance rate of 41% for proposal not involving a change in order with the first differenced filter, the numbers for the HP filtered series were 5% and 8% respectively. While Gelman, Roberts, and Gilks (1996), for example, provide conditions under which the optimal scaling in standard MCMC algorithms lies between 23% and 44%, Ehlers and Brooks (2004, p. 4) point out that "there is no Euclidean structure between models to guide proposal choices" for RJMCMC and our rates are in line with the ranges presented by Brooks, Giudici, and Roberts (2003). Stephens's (2000) birth-death RJMCMC, applied in autoregressive settings by Vermaak, Andrieu, Doucet, and Godsill (2004) and Philippe (2006), and Al-Awadhi, Hurn, and Jennison's (2004) secondary Markov chain method, along the alternatives explored by Brooks, Giudici, and Roberts (2003) provide alternatives and extensions that could potentially improve the acceptance rates in our settings.

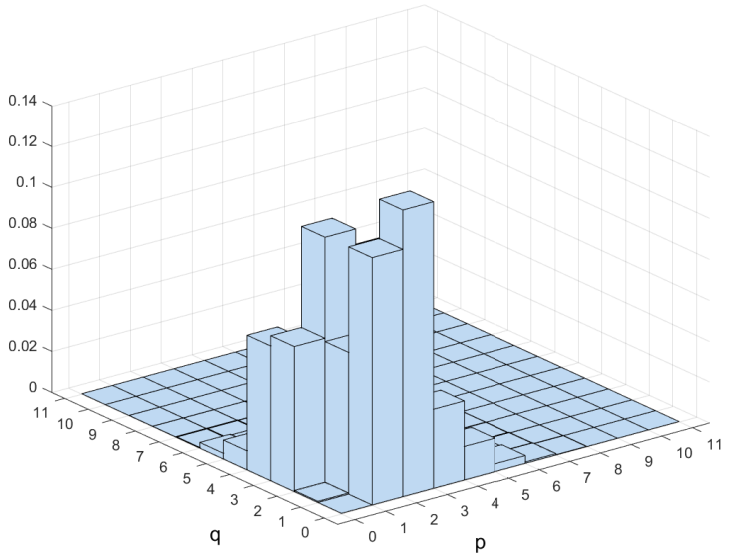

Figure 2: Posterior over the orders p, q for first differenced GDP data

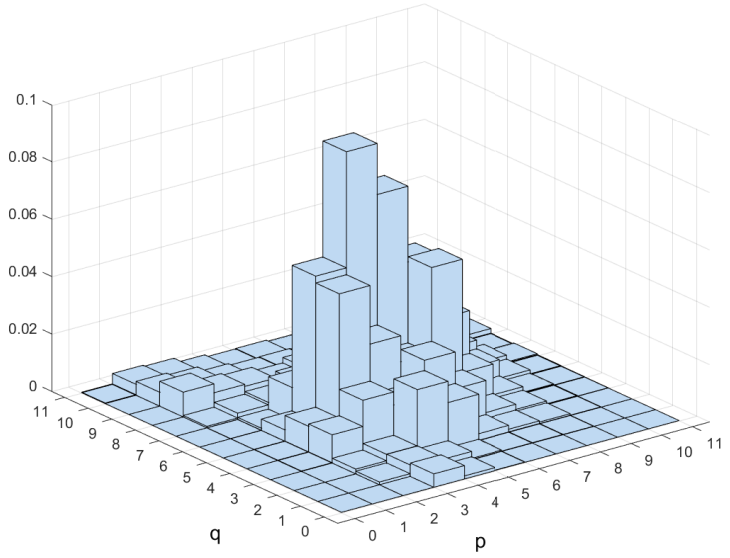

Figure 3: Posterior over the orders p, q for HP filtered GDP data

rior density spread over many more models than was the case with first differenced data. This is consistent with relatively high orders for the lag polynomials preferred at the posterior mode with many neighboring models mimicking the covariance structure of the model. That different filters can produce markedly different stationarized representations is well-known,[22] that the HP filtered data induces a posterior associated with higher order processes is consistent with the correlation functions presented by Cogley and Nason (1995, p. 203) of HP filtered random walks. We will return to the issue of filtering when we apply the method to estimate productivity shocks inside the DSGE model in the following sections. The model at our mode is an ARMA(4,5), with the posterior mean parameters conditional on posterior mode model given by

$$\begin{aligned} y_t =& 0.6027 y_{t-1} + 0.5304 y_{t-2} + 0.0861 y_{t-3} - 0.4196 y_{t-4} + \dots \\ &+ \epsilon_t + 0.3786 \epsilon_{t-1} - 0.2556 \epsilon_{t-2} - 0.5812 \epsilon_{t-3} - 0.2706 \epsilon_{t-4} - 0.2154 \epsilon_{t-5} \\ \epsilon_t &\sim N(0, 0.7551) \end{aligned}$$

3.3 Monte Carlo Setup

The Monte Carlo experiment is carried out by taking every 30,000th draw from the posterior for first differences and the second with every 70,000th draw from the posterior for HP filtered data, giving 100 different models each, and then for each generating 250 observations using the corresponding model and parameter values.

We implement RJMCMC by generating 1,500,000 draws from the posterior, discarding the first 1,000,000 as burn-in, and identifying the model at the mode in (p, q). The first state of the chain was set to white noise with unit standard deviation, i.e. $p = q = 0$ where p denotes the autoregressive order, q the moving average order, and $\sigma = 1$. Our metric for model choice is in accordance with a $0-1$ loss function, selecting the model at the mode of the posterior distribution over (p, q). It should be noted that one of the strengths of the method is the ability to quantify posterior uncertainty over models directly, such that model uncertainty can be incorporated in the calculation of posterior credible sets over impulse responses, correlations structures, or the like, providing more than just a point estimate of the model order.

We compare the model choice of our method with the choices that follow from minimizing the Akaike Information Criterion (AIC), the corrected Akaike Information Criterion (AICC), and the

[22] See, e.g., King and Rebelo (1993), Cogley and Nason (1995), and Canova (1998).

Schwarz Criterion (SC).[23] These are defined as

$$\text{AIC} = 2k - 2\ln(\hat{\mathcal{L}}), \quad \text{AICC} = AIC + \frac{2k(k+1)}{n-k-1}, \quad \text{SC} = -2\ln(\hat{\mathcal{L}}) + k\ln(n)$$

with k being the number of model parameters and n the number of observations. $\hat{\mathcal{L}}$ denotes the maximized likelihood value of a model, i.e., for given ARMA orders p and q.

3.4 Results

We report the proportion of correctly identified models in table 2 and the RJMCMC method outperforms the set of traditional information criteria in all cases.

Method	First Differences	HP-Filter
RJMCMC	0.23	0.05
AIC	0.08	0.03
AICC	0.09	0.02
SC	0.18	0.01

Table 2: Proportion of correctly identified models in the Monte Carlo experiments

The posterior using HP filtered data generally implies higher order processes, which hampers all methods' ability to correctly identify the model. This follows as the autocorrelation structure of ARMA models of higher orders may be very close even if the orders of the lag polynomials differ and the likelihood is therefore rather flat across models. This was reflected likewise in the posterior distribution over models in the initial estimation itself, see figure (3). In contrast to standard metrics, however, RJMCMC enables the characterization of the resulting uncertainty regarding model selection choices and the posterior therefore provides the researcher with a tool to gauge the extent of model uncertainty.

Of course, the ability of the method to estimate the parameters of the model along with the order of the model is of importance. Figure 4 reports the recursive means of the parameter draws of the model parameters data conditional on the model being correctly identified (i.e., $p = 2$ and $q = 0$ for first differenced data) from one chain. These values clearly converge close to the values underlying the data generating process.

In conclusion, the RJMCMC method exhibits roughly the same or better performance as classical methods concerning order identification while providing a complete posterior distribution over

[23] Calculations for the three standard measures were carried out using the R package auto.arima.

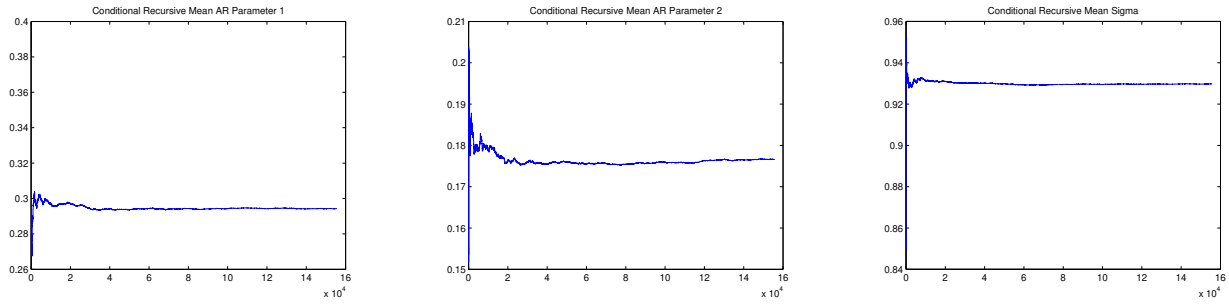

Figure 4: Recursive parameter means from the conditional posterior

parameters and model orders that can be used for the posterior analysis of statistics of interest. With first differenced and HP filter data, US post war GDP is best represented by an AR(2) and ARMA(4,5) respectively. We are interested in posterior statistics of DSGE models such as impulse responses and correlation structures and will now turn to a DSGE setting and apply the RJMCMC method there.

4 Neoclassical Growth Model

To examine how the RJMCMC method can be applied to a DSGE model, we consider Hansen's (1985) specification of the neoclassical growth model. In this simple model, the social planner's problem is to maximize the discounted lifetime expected utility of a representative household given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[ln\left(c_t\right) + \psi ln\left(1 - l_t\right) \right], \ 0 < \beta < 1 \tag{12}$$

with c_t representing consumption and l_t hours; $\beta \in (0,1)$ is the subjective discount factor of the household and ψ weights the utility of leisure $1 - l_t$ in the household's utility function. The social planner faces the resource constraint

$$c_t + i_t = y_t \tag{13}$$

where investment i_t contributes to the accumulation of capital k_t through

$$k_t = (1 - \delta) k_{t-1} + i_t \tag{14}$$

with the depreciation rate δ and where production y_t is neoclassical and given by

$$y_t = e^{z_t} k_{t-1}^{\alpha} l_t^{1-\alpha} \tag{15}$$

with z_t being stationary stochastic productivity. Hansen (1985) assumed a highly autocorrelated AR(1) process—with the autoregressive parameter set to 0.95—following Kydland and Prescott (1982). Relaxing this assumption will be the focus of our investigation in what follows, so we leave it otherwise unspecified for now.

The first order conditions of the social planner's problem are given by

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} \left(1 - \delta + \alpha e^{z_{t+1}} \left(\frac{l_{t+1}}{k_t} \right)^{1-\alpha} \right) \right] \tag{16}$$

$$\frac{\psi}{1 - l_t} = \frac{1}{c_t} (1 - \alpha) e^{z_t} \left(\frac{k_{t-1}}{l_t} \right)^{\alpha} \tag{17}$$

An equilibrium is defined by the equations (13) through (17) along with a specification for the stochastic productivity process z_t.

$\overline{L}$	$\frac{1}{3}$	Steady state employment 1/3 of total time endowment
α	0.36	Capital share
δ	0.025	Depreciation rate for capital
$\overline{R}$	1.01	One percent real interest rate per quarter

Table 3: Model calibration

In this exercise, we will take the parameters of Hansen's (1985) calibration of all parameters outside the specification of the stochastic productivity process z_t as given. This will allow us to concentrate on the contribution of the RJMCMC algorithm in estimating the order and parameters of the exogenous process.[24] The calibrated parameters reported in table 3 deliver standard values for parameters, imposing , e.g., that about one third of agents' time endowment is spent in employment activities, capital contributes a little more than one third to production. As we will consider arbitrary ARMA processes for z_t, the model does not fit canonical DSGE linear problem statements, e.g., Klein (2000), which allow for straightforward calculation of the likelihood function. While we could redefine the model to include the entire state vector induced by the ARMA exogenous process as endogenous variables to bring the model into the canonical form, doing so would significantly increase the computation costs involved in the QZ decomposition for the state transition and the Sylvester equation for the impact matrix of shocks. We apply the method of Meyer-Gohde and

[24]Ultimately, we will estimate deep parameters along with the orders and parameters of exogenous processes. In this study, we engage in the intermediate step of holding the deep parameters constant to focus on the exploration of the order and parameters of exogenous processes, avoiding the assessment of the relative influences of priors regarding orders and deep parameters on the posterior.

Neuhoff (2015), which solves linear DSGE models with (vector) ARMA processes of arbitrary orders directly.

5 Estimation Results for the Neoclassical Growth Model Model

We carry out three exercises using the neoclassical growth model as presented above. First, in order to check whether the method could pick up the correct underlying process for a technology shock in this model, we generated 250 observations of synthetic data using the AR(1) process as reported by Hansen (1985) in his original study. Second, we estimate the order and parameters of the technology shock process for the model using US GDP data, treated with the HP filter as in Hansen's (1985) original study. Finally, we examine the robustness of our results to this choice of filter and first difference the data instead.

5.1 Priors and Proposals

The priors and proposals for the shock process orders and parameters are reported in table 4.[25]

Variable	Prior	Proposal
p	U(0,10)	LaplaceD(p,2.2)
q	U(0,10)	LaplaceD(q,2.2)
AR PAC	TN(0,0.25)	TN(PAC,0.0016)
MA PAC	TN(0,0.25)	TN(PAC,0.0016)
σ	IG(1,1)	TN(σ,0.0025)

Table 4: Priors and proposals for the RBC model estimation

The priors remain the same as in the Monte Carlo study, while the dispersion parameters of the proposals were tuned using short pilot runs to increase the efficiency of the RJMCMC algorithm.[26]

[25] Our priors and proposals parallel those of the monte carlo study; see footnote 20.

[26] As discussed in footnote 21, the usual range of desired acceptance rates does not apply in the transdimensional RJMCMC. Our acceptance rates overall and for within model moves fall in the 5% to 10% range.

5.2 Synthetic AR(1) Data

For this exercise we generated 250 realizations for the technology shock according to the AR(1) specification and calibration in Hansen (1985)

$$z_t = 0.95 z_{t-1} + \epsilon_t, \quad \sim (0, 0.712^2) \tag{18}$$

We then fed the resulting series for z_t into the linearized RBC model and applied our method to the resulting synthetic data on output, y_t. We evaluate the likelihood function recursively using the Kalman filter to generate 650,000 draws from the posterior, discarding the first 100,000 draws as burn in. Standard visual measures indicated convergence. Figure 5 shows the posterior distribution over the orders for the disturbance. The method places an overwhelming majority of the posterior weight on the AR(1) model—obviously correctly identifying the AR(1) data generating process for the productivity process with observations on output, y_t. Furthermore, the AR(1) parameter and standard deviation of the innovation were 0.959 and 0.762, respectively, at the posterior mean conditional on the AR(1) model having been selected.

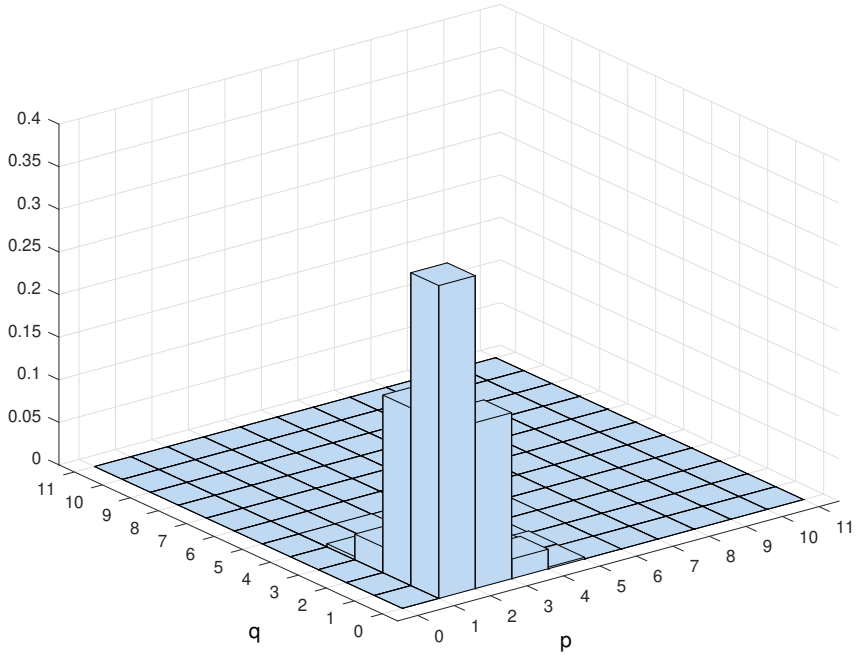

Figure 5: Posterior over the orders for the shock process, synthetic AR(1) data from (18)

This result gives us further confidence that, if the real world process for the productivity shock were AR(1), it would be correctly identified by the RJMCMC method we propose.

5.3 US GDP Data: Estimates

We now address what US postwar GDP data can reveal about the productivity shock in Hansen's (1985) model. We estimate the productivity shock process using HP filtered quarterly US GDP per capita as in Hansen (1985) taking his original calibration and value of 1600 for the smoothing parameter in the HP filter as given.[27] In applying the RJMCMC method introduced in section 2, we generated 4,000,000 draws per chain discarding the first 1,000.000 draws as burn in.[28] We apply the HP filter to the model when evaluating the likelihood,[29] thus treating the data and the model with the same filter.

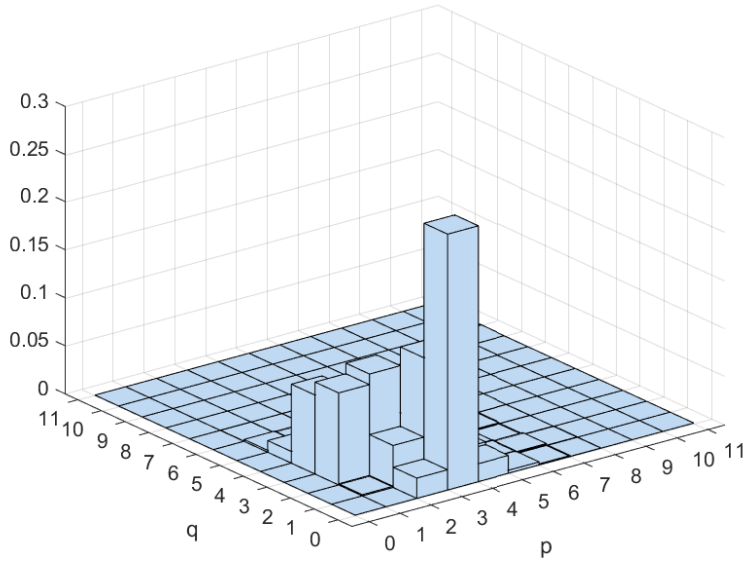

Figure 6: Posterior over the orders for the shock process, HP filtered data and model

Figure 6 shows the posterior over (p, q) for this exercise. The model at the mode is ARMA(3,0) and the baseline AR(1) specification of Hansen (1985) is clearly rejected. There is much more substantial uncertainty regarding the correct shock process than in the Monte Carlo exercises above. The prior posterior plots in figure (7) indicate that our results are not solely driven by our choice of priors, likewise confirmed by comparing the posteriors over orders in figure 6 to the implied priors

[27] See footnote 19 for details on the data series. We address robustness to the choice of filter in section 5.6.

[28] We initiated the five chains at $(0, 0)$, $(0, 10)$, $(10, 0)$, $(10, 10)$, and $(5, 5)$ for p and q.

[29] As the HP filter is a two-sided filter, we cannot use the Kalman filter to while applying the HP filter to evaluate the likelihood function the model. Thus, we treat the sample as one large draw from a multivariate normal distribution, where we calculate the sequence of HP filtered autocovariances spectrally. See Meyer-Gohde and Neuhoff (2015) for details on the computation.

in figure 1.

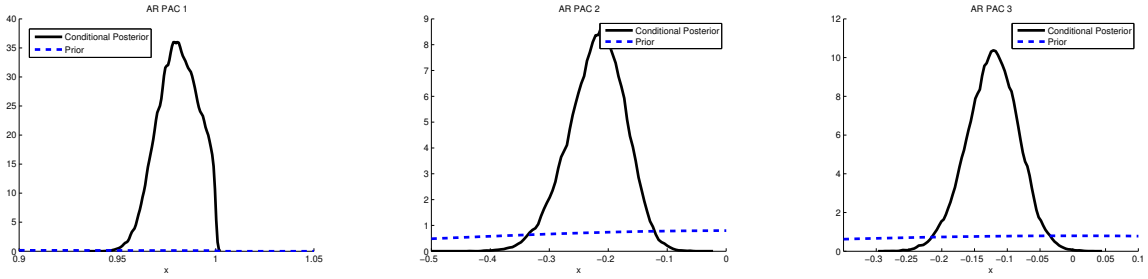

Figure 7: Priors and posteriors for partial autocorrelations, HP filtered data and model

Figure 8 reports recursive means of the first AR parameter for chains with differing initial states for the orders of the ARMA polynomial for the technology shock, calculated both conditional on the model at the mode of the posterior as well as unconditional means. Inspection suggests that all three chains have converged,[30] as do the posterior statistics, such as impulse responses, that we will examine.

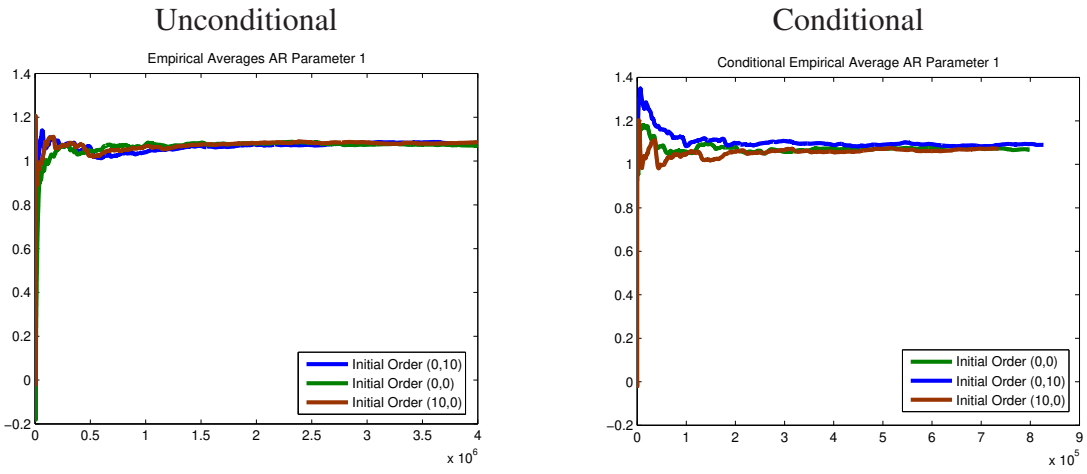

Figure 8: Convergence diagnostics, HP filtered data and model

Table 5 reports point estimates for the shock process parameters taken from the posterior distribution conditional on $(p, q) = (3, 0)$; the standard deviations for the posterior means are also reported. Additionally, the first two autocorrelations of the exogenous process, z_t, implied by these point estimates are given. The first autocorrelation is higher than, though consistent with, the choice

[30] It is not clear, however, whether these standard graphical or other formal measures of convergence, e.g., Brooks and Gelman (1998), apply without adaptation in transdimensional analyses, see e.g., Fan and Sisson (2011).

of Hansen (1985) following Kydland and Prescott (1982) to model the technology process with a near unit root.

Parameter	Mean	Median	Hansen (1985)
AR(1)	1.1689	1.1681	0.95
	(0.04)		
AR(2)	-0.0732	-0.0725	N/A
	(0.06)		
AR(3)	-0.1224	-0.1215	N/A
	(0.04)		
σ	0.5873	0.5733	0.712
	(0.08)		
$\rho(1)$	0.9804	0.9810	0.95
$\rho(2)$	0.9528	0.9542	0.9025

Table 5: Posterior point estimates and autocorrelations, HP filtered data and model along with Hansen's (1985) original AR(1) specification

5.4 US GDP Data: Correlation Structure

We now examine the variance and correlation structures implied by our posteriors and compare these with the data and the statistics implied by our baseline AR(1) model implied by Hansen (1985).[31] The posterior matches the structure of the second moments of output quite well, with the role of the prior becoming relatively more important for higher order correlations. As we estimated with real per capita GDP data, this is reassuring and indicates that the procedure does indeed provide a substantial improvement in fit.

Data	Hansen	Posterior Mode Model	Posterior Mode	90% Posterior Credible Set
2.8491	3.2574	2.8332	2.8182	2.1074 — 4.0965

Table 6: Standard deviation of output, in %, HP filtered data and model

The standard deviations of output are listed in table 6. Both the standard deviation of model at the posterior mode of the ARMA order and its associated parameter space as well as the posterior mode of the standard deviations across orders line up very close to the statistic in the data, whereas the statistic of Hansen (1985) shows greater a difference from the value in the data. The 80% posterior

[31] Following Hansen (1985), we calculate the second moments for his model using an HP filtered (with the smoothing parameter, λ, set to 1600) version of model.

credible set shows the extent of posterior uncertainty, which here is great enough to encompass all the point values reported.

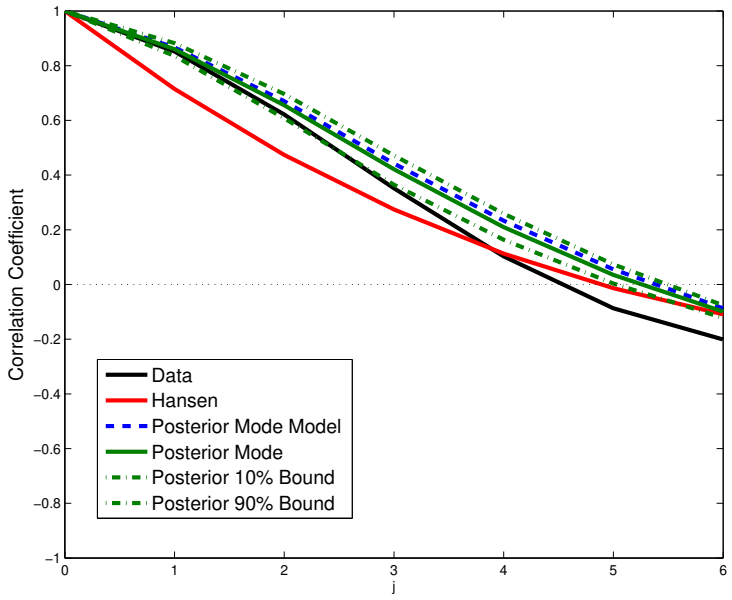

Figure 9: Comparison of autocorrelations of output, HP filtered data and model

The first six autocorrelations tell a more certain story, however, and can be found in figure 9. Again, both the autocorrelations of the model at the posterior mode of the ARMA order and its associated parameter space as well as the posterior mode of the autocorrelations match the statistic in the data very closely. The AR(1) structure imposed by Hansen (1985) forces a compromise, with the initial autocorrelation being somewhat lower and the later values somewhat higher than in the data.

The fit as implied by the point estimates of our posterior with respect to our observable series output is reassuring in that our application of the RJMCMC method is successfully doing what it should. With a mean zero normally distributed process, the second moments describe the stochastic properties of the process and our posterior brings the second moments of output from the RBC model closer to the data by selecting appropriate ARMA processes.

5.5 US GDP Data: Impulse Responses

With a posterior distribution over both models—i.e., orders p and q—and their parameters for the ARMA technology process, we plot impulse responses taking posterior uncertainty about the model into account. In the presence of MA components, this requires us to take a stand on which covariance equivalent representation we choose.[32] We will first examine the invertible or fundamental impulse responses associated with the posterior distribution. Then, we will allow the possibility of nonfundamental representations by sampling with a noninformative prior from the admissible (i.e., real valued) covariance equivalent representations and examine the resulting impulse responses.

In figure 10, we plot the impulse responses to a one standard deviation technology shock. We plot the invertible impulse associated with the model at the posterior mode of the ARMA order and parameter space against the pointwise posteriors (mode and 80% credible set) over all impulse responses weighted by posterior probabilities. To guarantee invertibility, we sample from the inverse partial autocorrelations analogously to our sampling from the partial autocorrelations for the AR components that guarantees stationarity. We also include the impulse response with Hansen's (1985) AR(1) technology assumption in the plot. The data driven selection of the specification of the shock process implies a different dynamic behavior for many of the model's endogenous variables compared to Hansen's (1985) AR(1) specification. While our procedure confirms the consumption smoothing and time to build responses of consumption and capital, we identify a sluggish response in all endogenous variables, a salient feature of the data identified in many empirical studies; e.g., Cogley and Nason (1995), who identify a hump shaped response of output to transitory technology shocks using both an SVAR and a VEC model. In essence, the sluggishness of output in the data that is captured by frictions in more sophisticated models, see especially Sims (1998) for an early assessment, is relegated to the exogenous process by our procedure.

We now move beyond imposing fundamentalness in the sampled MA components. In admitting nonfundamental MA representations, we acknowledge invertibility issues revisited in VAR contexts by, e.g., Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007) that imply that the covariance structure associated with our posterior distribution potentially implies several possible different structural representations. For an invertible or fundamental moving average representation,

[32] See Lippi and Reichlin (1994), Fernández-Villaverde, Rubio-Ramírez, Sargent, and Watson (2007), and Alessi, Barigozzi, and Capasso (2011) for more on different MA representations in macroeconomic modeling.

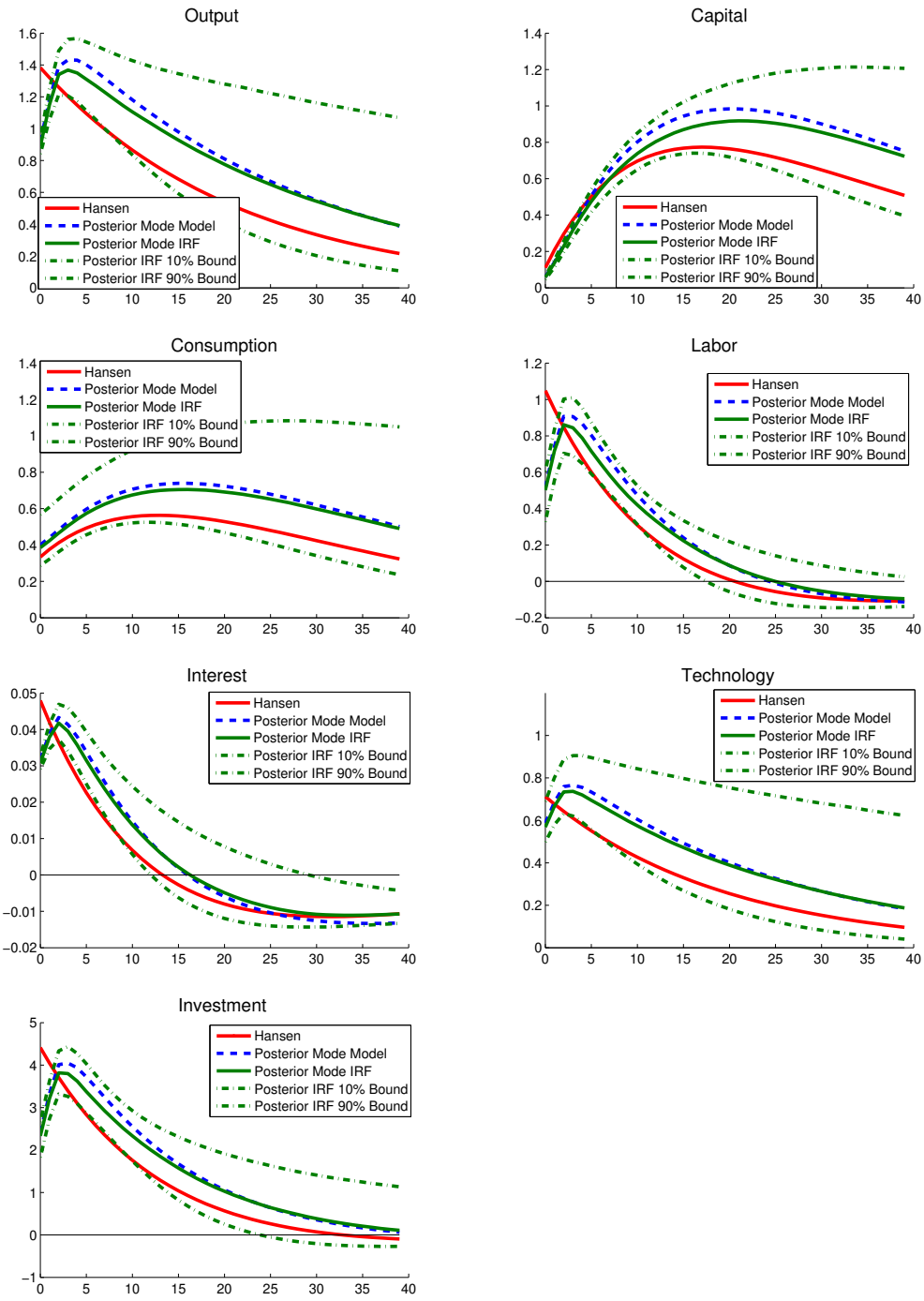

Figure 10: Impulse responses to a one standard deviation technology shock
Invertibility of MA components imposed
HP filtered data and model along with Hansen's (1985) original AR(1) specification

the roots, λ_{q_i}, of the MA polynomial

$$\gamma_i(\lambda) \doteq 1 + \gamma_{i,1}L \ldots + \gamma_{i,q}L^{q_i} \tag{19}$$

must all be contained outside the unit circle; that is, there exists no λ such that $\gamma_i(\lambda) = 0$ where $|\lambda| \geq 1$.[33] We follow Lippi and Reichlin's (1994) root-flipping procedure and propose the following algorithm to sample from admissible covariance equivalent representations assuming uninformative priors over the different representations.

Sampling From Admissible Covariance Equivalent Representations

1. For a given draw of order $q > 0$ for the MA component of the exogenous process, factor the MA polynomial as
$$1 + \gamma_{i,1}L \ldots + \gamma_{i,q}L^{q_i} = (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots\left(1 - \lambda_{q_i}L\right) \tag{20}$$
2. Enumerate all possible combinations of root flips. I.e., discard any combination that would flip only one of a complex conjugate pair of roots (as flipping only one of a pair of complex conjugates would lead to imaginary moving average parameters that we rule out on economic grounds), let $\tilde{n}$ denote this number of admissible combinations of root flips
3. Draw an integer $n \in \{0, 1, \ldots, \tilde{n}\}$ from a uniform distribution
4. Flip the roots according to the combination enumerated with n, where a draw of 0 indicates that no root is flipped (i.e., the invertible or fundamental representation is drawn.

For example, if $n = 10$ is drawn and the number 10 was associated with flipping roots λ_2 and λ_3, the MA polynomial for calculating impulse responses becomes

$$\gamma_i(L) = (-\lambda_2)(-\lambda_3)\left(1 - \frac{1}{\lambda_2}L\right)\left(1 - \frac{1}{\lambda_3}L\right)(1 - \lambda_1 L)(1 - \lambda_4 L)\ldots\left(1 - \lambda_{q_i}L\right) \tag{21}$$

Drawing the covariance equivalent representation from a uniform distribution over all admissible covariance equivalent representations puts equal weight on each admissible representation, reflecting our flat prior across the different representations over which DSGE theory is noninformative.

Figure 11 contains the pointwise posteriors (mode and 80% credible set) over all impulse responses weighted by posterior probabilities and drawn, potentially, from nonfundamental covariance equivalent representations as outlined above. We plot these pointwise posteriors against the invertible representation of the model at the posterior mode over ARMA orders and their parameter values, as well as against the impulse response with Hansen's (1985) AR(1) technology assumption. The admission of non-fundamental representations increases our uncertainty over the dynamic response of variables to a technology innovation, spreading the bounds of the 80% credible sets apart. Most of this spread is downward so that the number of periods for which the 80% credible set covers

[33] See, e.g., Hamilton (1994, p. 67).

exclusively positive responses to a technology shock is greatly reduced.

Admitting non-fundamental moving average representations places a negative response of hours to a positive technology shock is contained in the credible set. Hence even this simplest real business cycle model with an estimated technology shock process can recreate this stylized observation of Galí (1999) and Francis and Ramey (2005). The conclusion, therefore, that the stochastic growth model is unable to generate this response to technology shocks would require a strong prior against the noninvertible moving average representations, e.g., against news and policy announcement shocks. Nonetheless, the majority of the posterior mass still lies in a region where the response of hours to technology is conventional, in line with the results in Chari, Kehoe, and McGrattan (2008) and Uhlig (2004).

In sum, the posterior mode model and the posterior distribution over impulse responses, both fundamental and admitting the possibility of non-fundamental moving average representations, are markedly different than those implied by the AR(1) assumption in Hansen's (1985) original study. The data clearly favors hump-shaped impulse responses and cannot rule out a drop in hours in response to a positive technology shock.

5.6 Robustness to Data Filter

We now address the sensitivity of our results to the choice of filter, which is particularly important considering the starky different results we obtained with our ARMA estimates of GDP. We choose the first difference filter as alternative to the HP filter used above, following our choice in section 3. This filter is one sided so we return to using the Kalman filter to evaluate the likelihood of the filtered data using the filtered model.

The posterior distribution over orders (p, q) can be found in figure 12. The mode model is an ARMA(3,0), just as was the case above with the HP filter. Furthermore, the entire posterior across models in figure 12 is remarkably similar to the posterior using the HP filter, see figure 6. The posterior mean parameter estimates conditional on the ARMA(3,0) model with the first difference filter along with their standard deviations are juxtaposed with those with the HP filter in table 7. The extent of agreement is again striking.

Our RJMCMC estimates of exogenous processes in DSGE models do not appear to depend significantly on the choice of filter. However, this requires both the data and the model to be treated with the same filter, otherwise filter dependence as we found in section 3 would seem likely.

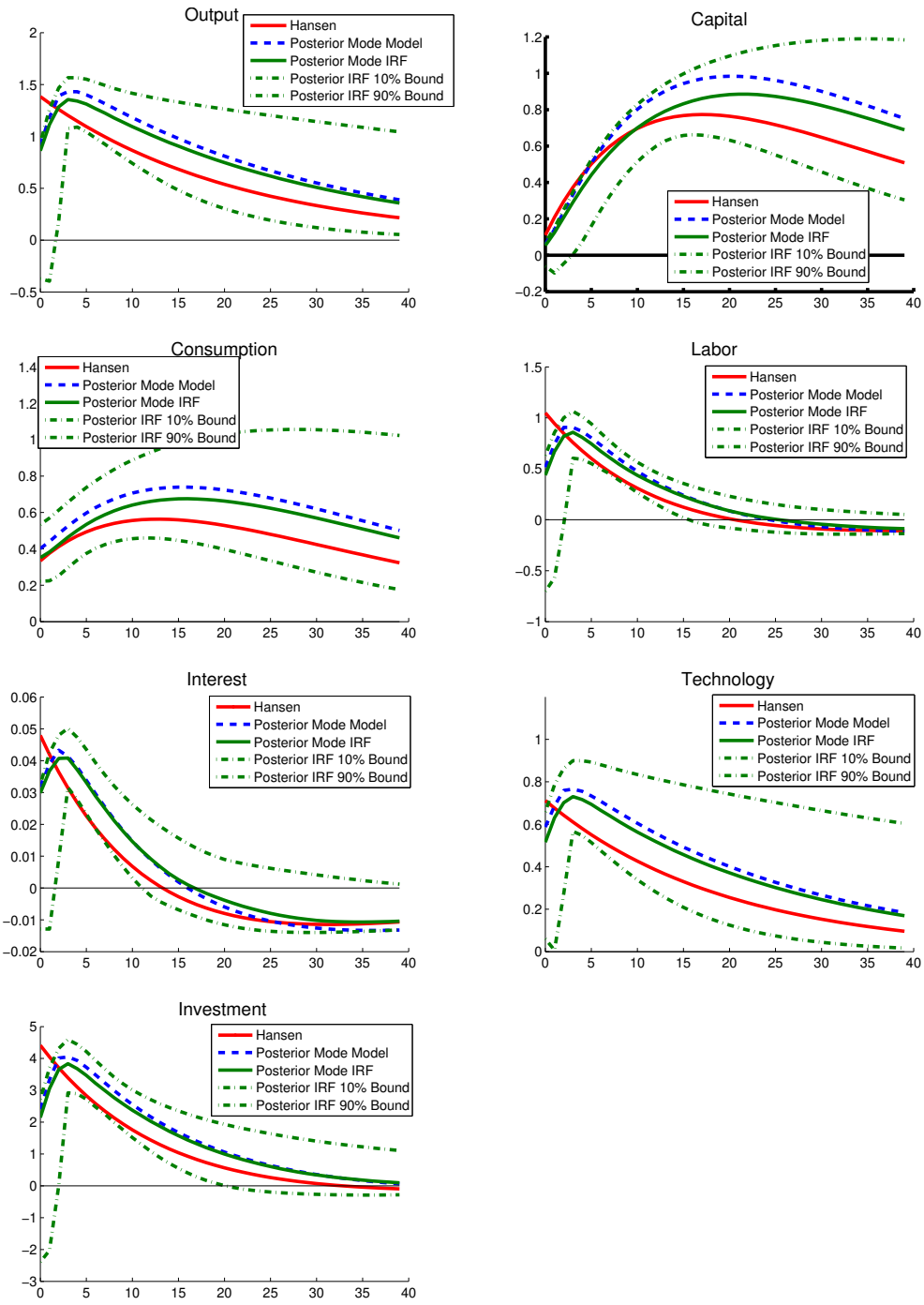

Figure 11: Impulse responses to a one standard deviation technology shock
Invertibility of MA components ***not*** imposed
HP filtered data and model along with Hansen's (1985) original AR(1) specification

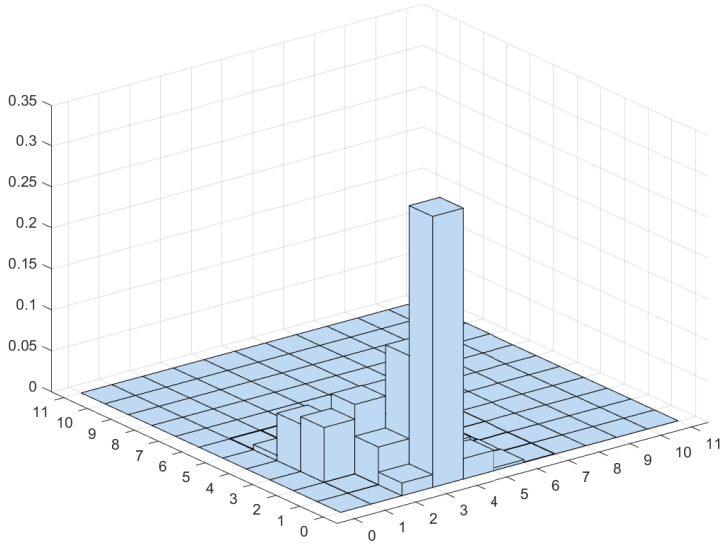

Figure 12: Posterior over the orders for the shock process, first differenced data and model

Parameter	First Differenced	HP Filtered
AR(1)	1.175	1.1689
	(0.04)	(0.04)
AR(2)	-0.0721	-0.0732
	(0,06)	(0.06)
AR(3)	-0.1289	-0.1224
	(0.04)	(0.04)
σ	0.59581	0.5873
	(0.07)	(0.08)

Table 7: Posterior mean estimates, first differenced vs. HP filter

6 Conclusion

We present a novel approach to addressing misspecification in DSGE models. Theory generally provides no guidance on the order of the exogenous processes and DSGE analyses seldom if ever estimate this order—the usual choice of the AR(1) structure on exogenous processes often lacks empirical support. We relax the assumptions placed on the structure of exogenous processes and estimate generalized ARMA(p,q) processes of unknown orders. Our method treats the ARMA orders of shock processes as additional parameters to be estimated, enabling the researcher to identify those shock process structures which bring the model closer to the data.

Our ARMA estimates of US post war GDP vary with the filter chosen and we demonstrate that the RJMCMC method compares favorably with traditional methods of order identification. Turning to the estimation of ARMA process for productivity shocks in the neoclassical growth model, we find compelling evidence for higher order processes, in contrast to the standard AR(1) assumption, and find that this result is robust to the choice of HP or first difference filter. Our posterior impulse responses are markedly different than those generated under the original calibration, with the higher order processes we estimate clearly identifying hump-shaped impulse responses hump-shaped responses for all endogenous variables including output. When taking an agnostic stance regarding the invertibility of the MA polynomials in our posterior, we cannot rule out a drop in hours in response to a positive technology shock.

The method allows for the quantification of posterior model uncertainty along with the posterior parameter uncertainty in standard DSGE Bayesian estimations. In future research will ultimately enable the analysis of a joint posterior over different specifications of the exogenous processes including their parameters as well as parameters of the model. By incorporating model uncertainty, the posterior impulse responses identified by our method offer an additional method of identifying empirical structural responses that use the entire model for identification and not just a subset of its short and/or long run restrictions. As noted in Del Negro and Schorfheide (2009), if one interprets the richer shock structure preferred by our method as a structural means of controlling for misspecification, the generalized shocks can simultaneously improve the accuracy of policy experiments and improve the fit of the model.

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A Appendices

A.1 Conventional Metropolis-Hastings Samplers

Markov Chain Monte Carlo (MCMC) methods in general provide samples from some probability distribution of interest by constructing a Markov chain whose stationary distribution is this distribution of interest. A Markov chain with the sequence of states $\varsigma_1, \varsigma_2, \ldots$ is specified in terms of the distribution for the initial state ς_1 and the transition kernel $K(\cdot)$ that provides the conditional distribution of a state ς_{i+1} given the current state ς_i. That is, the probability that ς_{i+1} is in some set $\mathcal{A} \subseteq \mathbb{R}^d$ given that the current state of the chain is ς_i is given by

$$K(\varsigma, \mathcal{A}) = P(\varsigma_{i+1} \in \mathcal{A} | \varsigma_i = \varsigma) \tag{A-1}$$

A distribution π is invariant for some Markov chain if the transition kernel of the chain satisfies

$$\int K(\varsigma, \mathcal{A}) \pi(\varsigma) d\varsigma = \int_{\mathcal{A}} \pi(\varsigma) d\varsigma \tag{A-2}$$

for all subsets $\mathcal{A}$ of the state space. The task in MCMC is to construct a kernel such that the distribution of interest π is invariant with respect to the Markov chain defined by $K()$. The expression in (A-2), however, is not practically useful for the construction of an appropriate kernel, as verifying (A-2) would involve integration over the unknown distribution π being sought.

One widely used approach to overcome this hurdle are Metropolis-Hastings samplers:[34] accept-reject samplers for which proposals for a new state of the chain are drawn from some distribution γ to be chosen by the researcher and then accepted with an appropriately derived probability α. Here, the stronger condition of reversibility or detailed balance is imposed, which guarantees that π is invariant for the Markov chain. This condition holds if a sequence of two states (ς, ς') has the same distribution as the reversed subchain (ς', ς) whenever $\varsigma, \varsigma' \sim \pi$. I.e., if

$$\int_{\mathcal{A}} \pi(\varsigma) K(\varsigma, \mathcal{B}) d\varsigma = \int_{\mathcal{B}} \pi(\varsigma') K(\varsigma', \mathcal{A}) d\varsigma' \tag{A-3}$$

for all subsets $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$. Condition (A-3) is more easily verified and can thus provide a starting point for the construction of a sampler.

A general Metropolis-Hastings algorithm can be written as follows: Let again ς denote a state of the Markov chain, in the case of Bayesian inference in the context of model estimation, the state is just the vector of model parameters and the distribution of interest is the posterior distribution

$$\pi(\varsigma) \propto \mathcal{L}(\varsigma) \rho(\varsigma) \tag{A-4}$$

[34] Laid out in Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller (1953) and generalized in Hastings (1970).

where ς denotes the vector of model parameters, $\mathcal{L}$ is the likelihood of the data given the model and its parameters and ρ is the prior over the model parameters. To obtain N samples from the posterior distribution, the following algorithm is run

Metropolis-Hastings

1. Set the (arbitrary) initial state ς_0 of the Markov chain
2. For $i = 1$ to N
   (a) Set $\varsigma = \varsigma_{i-1}$
   (b) Propose a new state from some proposal distribution $\gamma(\varsigma'|\varsigma)$
   (c) Accept draw with probability
   $$\alpha(\varsigma, \varsigma') = \min(1, \chi)$$
   with
   $$\chi = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \times \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \times \underbrace{\frac{\gamma(\varsigma|\varsigma')}{\gamma(\varsigma'|\varsigma)}}_{\text{Proposal Ratio}}$$
   (d) If the draw is accepted set $\varsigma_i = \varsigma'$. If the draw is rejected set $\varsigma_i = \varsigma$

This algorithm defines a transition kernel such that the Markov chain has the desired invariant distribution. The sequence of states of the chain is then a sample from this distribution of interest. The acceptance probability α corrects for differences between the proposal distribution γ and the distribution of interest.[35]

The kernel in the above is given by

$$K(\varsigma, \mathcal{B}) = \underbrace{\int_{\mathcal{B}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma, \varsigma')d\varsigma'}_{\text{Probability of moving to set } \mathcal{B}} + \underbrace{\left[1 - \int_{\mathcal{B}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma, \varsigma')d\varsigma'\right] \mathbb{1}_{\varsigma}}_{\text{Probability of rejecting the move and } \varsigma \in \mathcal{B}} \tag{A-5}$$

where $\mathbb{1}_{\varsigma} = 1$ if $\varsigma \in \mathcal{B}$ and zero otherwise, giving the probability of moving to some subset $\mathcal{B}$ of the parameter space conditional on the chain currently being at ς. The crux when constructing the kernel is to define the appropriate acceptance probability α and the proposal distribution γ so as to satisfy the detailed balance condition and thereby guarantee the convergence of the Markov chain to the desired probability distribution. Indeed, plugging in the formulation of the kernel from (A-5) into (A-3) gives an expression from which, given the proposal distribution γ the appropriate acceptance probability α can be readily derived using Peskun's (1973) recipe.

[35] Note that in the case of a standard random walk Metropolis-Hastings sampler with symmetric proposals, i.e. a Metropolis sampler, the proposal ratio reduces to one.

A.2 Detailed Derivation of Inflated Proposal Mapping

To choose an appropriate mapping $g_{pp'}$, it is useful to break the mapping into two parts according to the desired parameters P_p and the auxiliary parameters u. The mapping $g_{pp'}$ is given by

$$(P^{p'}, u') = g_{pp'}(P^p, u) = (g_{1pp'}(P^p, u), g_{2pp'}(P^p, u)) \tag{A-6}$$

and its inverse

$$(P^p, u) = g^{-1}_{pp'}(P^{p'}, u') = g_{p'p}(P^{p'}, u') = (g_{1p'p}(P^{p'}, u'), g_{2p'p}(P^{p'}, u') \tag{A-7}$$

Start with $g_{1pp'}$. Suppose now that the current state of the Markov chain is at $\varsigma = (p, P^p)$. Now with probability $\gamma_p(p'|p)$, a move to the model with order p' is proposed. Conditional on this proposal, we draw u from some proposal distribution $\gamma_{pp'}(u)$. Then, we introduce a deterministic mapping $g_{1pp'}$ that maps the current state and the auxiliary proposal u to the proposed new state such that $(p', P^{p'}) = (p', g_{1pp'}(P^p, u))$. Note that u is not part of the state of the chain.

Additionally, we have to find $g_{2pp'}$. In order to be able to easily verify adherence to detailed balance for a move from a state (p, P^p) to $(p', P^{p'}) = (p', g_{1pp'}(P^p, u))$ the vectors of Markov chain states and the random auxiliary proposal variables (P^p, u) and $(P^{p'}, u')$ must be of equal dimension and requiring $g_{pp'}$ to be a differentiable bijection lets us use a simple change-of-variables in the detailed balance equation. I.e., the kernel of the chain is now defined in terms of the auxiliary variable u together with the model indicator and the parameter vectors.

Armed with this structure it is now straightforward to derive the appropriate acceptance probability. The detailed balance condition holds if[36]

$$\int_{\mathcal{A}_p} \pi(p|y)\,\pi(P^p|p,y)\,Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int_{\mathcal{B}_{p'}} \pi(p'|y)\,\pi\left(P^{p'}|p',y\right) Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} \tag{A-8}$$

for all subsets $\mathcal{A}_p$ and $\mathcal{B}_{p'}$ of the parameter spaces associated with autoregressive polynomials of order p and p' respectively. The posterior distribution $\pi(\varsigma|y)$ is factorized as $\pi(\varsigma|y) = \pi(p|y)\pi(P^p|p,y)$ and

$$\begin{aligned} Q\left(\varsigma, \mathcal{B}_{p'}\right) &= \int_{\mathcal{B}_{p'}} \gamma(\varsigma'|\varsigma)\alpha(\varsigma,\varsigma')d\varsigma' \\ &= \gamma_p(p'|p) \int \mathbb{1}(g_{1pp'}(P^p,u) \in \mathcal{B}_{p'})\alpha_{pp'}(P^p, g_{1pp'}(P^p,u)\gamma_{pp'}(P^p,u)du \end{aligned}$$

The left hand side of (A-8) is then

$$\int_{\mathcal{A}_p} \pi(\varsigma|y)\,Q\left(\varsigma, \mathcal{B}_{p'}\right) dP^p = \int\int \mathbb{1}(P^p \in \mathcal{A}_p, g_{1pp'}(P^p,u) \in \mathcal{B}_{p'})\pi(p|y)\,\pi(P^p|p,y) \times \tag{A-9}$$

$$\gamma_p(p'|p)\alpha_{pp'}(P^p, g_{1pp'}(P^p,u)\gamma_{pp'}(P^p,u)dP^p du \tag{A-10}$$

[36] See also Waagepetersen and Sorensen (2001).

and the right hand side reads

$$\int_{\mathcal{B}_{p'}} \pi(\varsigma'|y)\, Q\left(\varsigma', \mathcal{A}_p\right) dP^{p'} = \int\int \mathbb{1}(P^{p'} \in \mathcal{B}_{p'}, g_{1p'p}(P^{p'}, u') \in \mathcal{A}_p)\pi(p'|y)\, \pi\left(P^{p'}|p', y\right) \times \tag{A-11}$$

$$\gamma_p(p|p')\alpha_{p'p}(P^{p'}, g_{1p'p}(P^{p'}, u'))\gamma_{p'p}(P^{p'}, u')dP^{p'}du' \tag{A-12}$$

where $\gamma(\varsigma'|\varsigma)$ is again factorized as $\gamma_p(p|p)\gamma_{pp'}(P^p, u)$. The fact that $g_{pp'}$ is a differentiable bijection together with the dimension matching conditions enables a change of variable in (A-11) leading to

$$\int\int 1(g_{1pp'}(P^p, u) \in \mathcal{B}_{p'}, P^p \in \mathcal{A}_p)\pi(p'|y)\, \pi\left(g_{1pp'}(P^p, u)|p', y\right)\gamma_p(p|p')$$

$$\times\alpha_{p'p}(g_{1pp'}(P^p, u), P^p)\gamma_{p'p}(g_{1pp'}(P^p, u), g_{2pp'}(P^p, u))|g'_{pp'}(P^p, u)|dP^p du \tag{A-13}$$

where $dP^{p'}du' = |g'_{pp'}(P^p, u)|dP^p du$ and $|g'_{pp'}(P^p, u)|$ is the determinant of the Jacobian of $g_{pp'}$.

By inspection of (A-9) and (A-13), the reversibility condition (A-8) is satisfied if

$$\pi(p|y)\, \pi(P^p|p, y)\, \gamma_p(p'|p)\alpha_{pp'}(P^p, g_{1pp'}(P^p, u))\gamma_{pp'}(P^p, u) =$$

$$\pi(p'|y)\, \pi\left(g_{1pp'}(P^p, u)|p', y\right)\gamma_p(p|p')\alpha_{p'p}(g_{1pp'}(P^p, u), P^p)\times$$

$$\gamma_{p'p}(g_{1pp'}(P^p, u), g_{2pp'}(P^p, u))|g'_{pp'}(P^p, u)| \tag{A-14}$$

Choosing the acceptance probability as large as possible, we have

$$\alpha_{pp'} = \min\left(1, \chi_{pp'}(\varsigma, \varsigma')\right) \tag{A-15}$$

with

$$\chi_{pp'}(\varsigma, \varsigma') = \underbrace{\frac{\mathcal{L}(\varsigma')}{\mathcal{L}(\varsigma)}}_{\text{Likelihood Ratio}} \underbrace{\frac{\rho(\varsigma')}{\rho(\varsigma)}}_{\text{Prior Ratio}} \underbrace{\frac{\gamma_p(p|p')\gamma_{p'p}(g_{pp'}(P^p, u))}{\gamma_p(p'|p)\gamma_{pp'}(P^p, u)}|g'_{pp'}(P^p, u)|}_{\text{Proposal Ratio}} \tag{A-16}$$

With our mapping $g_{pp'}$, in (4), $|g'_{pp'}(P^p, u)|$ is equal to one and (A-16) reduces to (9).[37]

A.3 Imposing Stationarity and Invertibility on ARMA(p,q) Sampling

To constrain sampling to these invertible and stationary regions of the parameters spaces of each model, we follow Barndorff-Nielsen and Schou (1973), Monahan (1984) and Jones (1987) and reparametrize the AR (and MA) polynomial in terms of its (inverse) partial autocorrelations (PACs). If the (inverse) partial autocorrelations are between -1 and 1 the process is (invertible) stationary.

First, we generalize the AR(p) model to an ARMA(p,q) as follows

$$y_t = P_1^{p,q} y_{t-1} + P_2^{p,q} y_{t-2} + \ldots + P_p^{p,q} y_{t-p} + \epsilon_t + Q_1^{p,q}\epsilon_{t-1} + \ldots + Q_q^{p,q}\epsilon_{t-q}, \quad \epsilon_t \sim \mathcal{N}\left(0, \sigma^2\right) \tag{A-17}$$

In order to recover the coefficients of the AR polynomials, the following algorithm is run

[37] The posterior π is here written factorized as the product of likelihood and prior $\mathcal{L}(\varsigma)\rho(\varsigma)$ for correspondence with the general formulation of the detailed balance condition (A-3).

Recovering AR Coefficients from PACs

1. Introduce $p^k = \left(p_1^{(k)}, \ldots, p_k^{(k)}\right), k = 1, \ldots, p$
2. Draw $r = r_1, \ldots, r_p,$ for $r_i \in (0, 1)$ partial autocorrelations
3. Set $p_1^{(1)} = r_1$
4. Run the recursion
$$p_i^{(k)} = p_i^{(k-1)} - r_k p_{k-i}^{(k-1)}, i = 1, \text{ for } \ldots, k-1$$
with $p_k^{(k)} = r_k$ for $k = 2, \ldots, p$
5. Set $P^p = p^{(p)}$

The MA coefficients are recovered analogously, where the inverse partial autocorrelations substitute for the partial autocorrelations, r_i, in the foregoing. Ultimately, instead of proposing AR(MA) parameters directly, (inverse) partial autocorrelations are proposed in their place from which the parameters are then recovered. This will obviously necessitate the formulation of priors over (inverse) partial autocorrelations instead of parameters.

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