Institute for Monetary and Financial Stability

ALEXANDER MEYER-GOHDE

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Institute for Monetary and Financial Stability
GOETHE UNIVERSITY FRANKFURT

WORKING PAPER SERIES NO. 207 (2024)

Institute for Monetary and Financial Stability
Goethe University Frankfurt
House of Finance
Theodor-W.-Adorno-Platz 3
D-60629 Frankfurt am Main
www.imfs-frankfurt.de | info@imfs-frankfurt.de

SOLVING AND ANALYZING DSGE MODELS IN THE FREQUENCY DOMAIN

ALEXANDER MEYER-GOHDE

ABSTRACT. I provide a solution method in the frequency domain for multivariate linear rational expectations models. The method works with the generalized Schur decomposition, providing a numerical implementation of the underlying analytic function solution methods suitable for standard DSGE estimation and analysis procedures. This approach generalizes the time-domain restriction of autoregressive-moving average exogenous driving forces to arbitrary covariance stationary processes. Applied to the standard New Keynesian model, I find that a Bayesian analysis favors a single parameter log harmonic function of the lag operator over the usual AR(1) assumption as it generates humped shaped autocorrelation patterns more consistent with the data.

JEL classification codes: C32, C62, C63, E17, E47

Keywords: DSGE; solution methods; spectral methods; Bayesian estimation; general exogenous processes

1. INTRODUCTION

This article continues the analysis of linear DSGE models in the frequency domain following Whiteman (1983). In contrast to standard time series methods that characterize solutions as bounded sequences, this approach seeks solutions as analytic functions on the unit disk. While the connection between these two via factorization and operator approaches has been established,[1] this connection for multivariate models is incomplete. Furthermore, the necessity of the frequency domain approach in some cases is not

I am grateful to Uwe Hassler and Mary Tzaawa-Krenzler for helpful comments and Michel Neumann for research assistance. Any and all errors are entirely my own. This research was supported by the DFG through grant nr. 465469938 "Numerical diagnostics and improvements for the solution of linear dynamic macroeconomic models".

Goethe-Universität Frankfurt and Institute for Monetary and Financial Stability, Chair of Financial Markets and Macroeconomics, Theodor-W.-Adorno-Platz 3, 60629 Frankfurt am Main, Germany Email: Author's Email, Author's Website, Github Repository

Date: August 25, 2024.

[1]See especially Taylor (1986, ch. 2.3).

appreciated by the literature - such as lag patterns beyond AR and MA representations. This paper attempts to close these gaps, deriving a frequency domain solution based on the familiar QZ or generalized Schur decomposition, provides a factorization of the multivariate lead-lag model and applies the methodology to solve, estimate, and analyze the canonical New Keynesian model with non ARMA driving forces. I find that a log harmonic function of the lag operator from the digital signal processing and systems theory literature is preferred by the data over the standard AR(1) assumption in DSGE modelling, being able to parsimoniously produce hump shaped dynamics in macroeconomic variables.

Standard DSGE methods today use state space, time domain methods to solve and estimate models, see Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016), with the solution usually being obtained via the QZ or generalized Schur decomposition, that separates the eigenspace into stable and unstable spaces with respect to the unit circle and constructs the solution using the former.[2] For the frequency domain perspective, Futia (1981) laid the groundwork by highlighting the importance of spectral properties in assessing the stability of equilibria in stationary linear models. Whiteman (1983) provided a practical guide to applying these methods across a wider range of models. Hansen and Sargent (1980, 1981) further integrated spectral analysis into the econometric estimation and prediction of dynamic models, using the Wiener-Kolmogorov prediction formula and factorizations. Taylor (1986) synthesized these contributions, connecting them for solving and analyzing rational expectations and DSGE models. Spectral approaches to solving DSGE models provides a framework for analyzing the existence, uniqueness, and stability of equilibria by leveraging the frequency domain to address challenges beyond the reach of state space methods such as infinite lags and leads and have seen a recent resurgence.[3] Al-Sadoon (2020) emphasizes that solutions to DSGE models form a finite-dimensional affine space in the frequency domain, which allows for precise characterization and distinction of solutions within this framework. Onatski's (2006) contribution, through the introduction of the winding number criterion, further advances this methodology

[2]Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot, 2011; Villemot, 2011), Gensys (Sims, 2001), Uhlig's Toolkit (Uhlig, 1999) and Solab (Klein, 2000) all follow this approach.

[3]The frequency domain perspective had not by any means disappeared, remaining a mainstay in imperfect information approaches such as Kasa's (2000) application to higher order beliefs, Kasa, Whiteman, and Walker's (2011) application to information revealing prices under heterogeneous information, Leeper and Walker's (2011) news and Leeper, Yang, and Walker's (2012) anticipation shocks in DGSE.

by providing a geometric criterion that links the spectral characteristics of a model to the existence and uniqueness of its equilibrium. This criterion determines whether a DSGE model will yield a unique solution, multiple solutions, or no solution, based on the behavior of its characteristic function in the complex plane, an approach adapted by Loisel (2022) to analyze solution uniqueness in broad set of models.[4] Building on these foundational ideas, Tan and Walker (2015) and Tan (2021) propose a frequency-domain framework for solving and estimating DSGE models,[5] demonstrating its applicability across a broad spectrum of economic scenarios, including linear and multivariate models.

Tan and Walker (2015) and Tan (2021) provide the closest references to this paper. Whiteman (1983) shows how continuing the analytic function on the unit disk that constitutes the solution to the DSGE model in the frequency domain over singularities on the disk can pin down missing initial conditions, i.e., determines the value of jump variables to put the system on the stable arm from a phase diagram perspective. The presence of the correct number of singularities is analogous to the Blanchard and Kahn (1980) conditions and is intuitively behind the analyses of Onatski's (2006) and Loisel (2022) and this continuation is the annihilation operator or plussing in the prediction formula of Hansen and Sargent (1980) and Hansen and Sargent (1981). Establishing the presence of and continuing an analytic function over a singularity is a nontrivial task in a multivariate setting. Tan and Walker (2015) and Tan (2021) diagonalize the model using the Smith normal form and then analyze the problem recursively. The Smith normal form, however, is numerically unstable, analogously to the Jordan normal form being a general invertible decomposition, see, e.g., Van Dooren (2004), and as such ought to be avoided in numerical analysis. Tan and Walker (2015) and Tan (2021) circumvent this problem to some extent by making avail of symbolic implementations of Smith normal form. It is unclear how scalable this is - particularly for policy institutions working with medium or large scale DSGE models this is a concern. I show how to triangularize a canonical DSGE model for finding and continuing the analytic solution functions over singularities

[4]In a similar vein, Meyer-Gohde and Tzaawa-Krenzler (2024) show that Mankiw and Reis's (2002) sticky information Phillips curve is recursive in the frequency domain and leverage this to provide analytic results on determinacy.

[5]Sala (2015) estimated DSGE models in the frequency domain. While the solution was obtained using standard time-domain techniques, the solution was then transformed to the frequency domain to enable the estimation of the model based on different frequency bands. Dück and Verona (2023) and Martins and Verona (2023) are two recent examples of analyses that examine frequency consequences, but not frequency solutions, of macroeconomic models.

using an orthogonal decomposition, namely the generalized Schur decomposition or QZ algorithm familiar to DSGE analysis in the time domain. This serves two purposes, firstly, to address the numerical or scalability obstacles of the Smith normal form approach of Tan and Walker (2015) and Tan (2021) and to provide researches with more familiar mathematical tools, the QZ algorithm, in the hopes that this makes the technique more approachable. Supporting this second purpose, the analysis here links the multivariate spectral approach to a multivariate factorization in lag and forward operators, the direct extension of Sargent's (1987, Ch. XIV) univariate factorizations.

I then demonstrates the practical application of the method by estimating and analyzing a New Keynesian model, specifically the loglinearized version described in Herbst and Schorfheide (2015). While traditional models, such as Smets and Wouters (2007), often rely on AR(1) processes for technology and government expenditures, this research expands the scope by introducing alternative specifications, including MA(1) and nonlinear (or rather non polynomial) processes like log and harmonic lag operators. These nonlinear processes, drawn from the systems and digital signal processing literature, capture autocorrelation patterns that cannot be analyzed using standard time-domain methods. Through a Bayesian analysis that compares the four specifications, each constrained to a single parameter for parsimony, I find that the log harmonic lag specification is most favored by the data and hence outperforms the standard AR(1) specification. This follows its ability to produce hump-shaped autocorrelation and impulse responses, a common characteristic in macroeconomic time series, as highlighted by Cogley and Nason (1995) and, particularly, by capturing the persistent effects of government expenditure shocks on output growth. This provides a compelling argument for moving beyond traditional AR(1) processes in macroeconomic modeling - to capture the more complex autocorrelation structures that yield a better empirical fit, a frequency domain solution approach as provided here is essential.

The remainder of the paper is organized as follows. In section 2, I layout the fundamentals of time series in the frequency domain and demonstrate how to solve scalar models, using Cauchy's residue theorem to continue the endogenous variable's rational transfer function over a singularity on the unit disk caused by its forward looking component. I then extend this approach to multivariate settings in section 4, connecting to factorization techniques and then providing a multivariate extension of the residue theorem approach via the generalized Schur decomposition to triangularize the model. In section 6, I review how the impulse responses and second moments can be recovered using inverse Fourier transforms, the latter enabling likelihood calculations for normally distributed variables. The techniques are applied to the estimation and analysis of a canonical New Keynesian model in section 7, estimating the model under four different assumptions on exogenous driving processes, two of which have non polynomial functions in the lag operator, making time domain techniques ill suited. Finally, section 8 concludes.

2. Time Series in the Frequency Domain

2.1. **Recursive Time Domain and Rational Transfer Function.** To lay out the analysis, I present an (incomplete) introduction of the relevant frequency domain properties for the analysis.[6] Whiteman (1983) assumes, and we follow, that solutions for y_t are sought in the space spanned by time-independent square-summable linear combinations of the process(es) fundamental for the driving process, that is H^2 or Hardy space.[7] Let ϵ_t be such a mean zero fundamental process with variance σ_ϵ^2. Then an H^2 solution for an endogenous variable, y_t, is of the form

$$y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j} \tag{1}$$

with $\sum_{j=0}^{\infty} y_j^2 < \infty$ and L the lag operator $Ly_t = y_{t-1}$.[8] Following, e.g., Sargent (1987, ch. XI) the Riesz-Fischer Theorem gives an equivalence (a one-to-one and onto transformation) between the space of squared summable sequences $\sum_{j=0}^{\infty} y_j^2 < \infty$ and the space of analytic functions in unit disk $y(z)$ corresponding to the z-transform of the sequence, $y(z) = \sum_{j=0}^{\infty} y_j z^j$.

Given a discrete series y_j its z-transform $y(z)$ is defined as

$$y(z) = \sum_{j=0}^{\infty} y_j z^j \tag{2}$$

where z is a complex variable, and the sum extends from 0 to infinity, following the convention used in Hamilton (1994, ch. 6) and Sargent (1987, ch. XI).[9] By evaluating

[6] See the appendix for a more complete representation theorem which we forgo here for expediency.

[7] See also Han, Tan, and Wu (2022) for a more detailed introduction.

[8] Note that we are abusing notation somewhat and choosing to use the same letter y to refer to a discrete time series, y_t, as well as that variable's transform function $y(z)$ or MA representation/response to a fundamental process j periods ago, y_j. This serves to save on the verbosity of notation, which might otherwise read $y_t = \sum_{j=0}^{\infty} \delta_j^y \epsilon_{t-j}$ following, e.g., Meyer-Gohde (2010).

[9] The discrete signal processing and systems theory literature works in negative exponents of z, see Oppenheim, Schafer, and Buck (1999, ch. 3) and Oppenheim, Willsky, and Nawab (1996, ch. 10). Al-Sadoon

the z-transform on the unit circle in the complex plane ($z = e^{-i\omega}$, where ω is the angular frequency and i the complex number $\sqrt{-1}$), we obtain the discrete-time Fourier transform

$$y(e^{-i\omega}) = \sum_{j=0}^{\infty} y_j e^{-i\omega j} \tag{3}$$

The connection between the autocovariance function and the Fourier transformation of the z-transform evaluated on the unit circle ($z = e^{-i\omega}$)

$$R_y(m) = \frac{\sigma_\epsilon^2}{2\pi} \int_{-\pi}^{\pi} \left| y(e^{-i\omega}) \right|^2 e^{im\omega} d\omega \tag{4}$$

and directly from the Riesz-Fischer theorem's transform pair,

$$y_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} y(e^{-i\omega}) e^{ji\omega} d\omega \tag{5}$$

These relationships allow us to analyze the temporal dependencies in a time series. By leveraging the z-transform and Fourier transform, along with the calculations of autocovariance and impulse responses (or moving average coefficients), we will uncover the frequency content and temporal dynamics of discrete-time series that are subject to sticky information.

3. Residues and Scalar Models

Having laid out the basic properties and paid specific attention to the scaling in the z domain property, we now turn to solving rational expectations models in the frequency domain following Whiteman (1983) - see also Taylor (1986, ch. 2.3) for an approachable introduction with direct comparisons to other methods.

Consider a backward and forward looking model in scalar y_t and w_t

$$aE_t y_{t+1} + by_t + cy_{t-1} + w_t = 0 \tag{6}$$

where w_t is an exogenous process with an $MA(\infty)$ representation $w_t = w(\mathscr{L})\epsilon_t$ in the innovation ϵ_t with $\mathscr{L}$ the lag operator $\mathscr{L}\epsilon_t = \epsilon_{t-1}$.

Starting with expectations, the Wiener-Kolmogorov prediction formula gives $E_t[y_{t+n}] = E_t\left[\sum_{j=0}^{\infty} y_j \epsilon_{t-j+n}\right] = \sum_{j=0}^{\infty} y_{j+n}\epsilon_{t-j}$. The Wiener-Kolmogorov prediction formula of "plussing" gives the frequency domain version

$$\mathcal{Z}\{E_t[x_{t+1}]\} = \left[\frac{x(z)}{z}\right]_+ = \frac{1}{z}(x(z) - x(0)) \tag{7}$$

(2020) follows this convention and interprets the operator being applied as the forward operator. I maintain the more familiar approach in working with the lag operator which results in the use of positive exponents in z.

where $_+$ is the annihilation operator, see Sargent (1987) and Hamilton (1994).

Applying a z-transform to (6), noting (7),

$$a\frac{1}{z}(y(z)-y_0)+by(z)+czy(z)+w(z)=0 \tag{8}$$

Rearranging allows me to reduce the solution to this model as

$$a(y(z)-y_0)+bzy(z)+cz^2y(z)+z=0 \Leftrightarrow (a+bz+z^2)y(z)=ay_0-zw(z) \tag{9}$$

$$(a-a(\lambda_1+\lambda_2)z+a\lambda_1\lambda_2z^2)y(z)=ay_0-z \Leftrightarrow (1-\lambda_1z)(1-\lambda_2z)y(z)=y_0-\frac{zw(z)}{a} \tag{10}$$

with the initial condition on y_0 to be determined.

I will require that $y(z)$ be analytic inside the unit disk to give us a stable process y_t causal in ϵ_t. Consider now the following possibilities. If $|\lambda_1|,|\lambda_2|<1$, then there is no singularity in $y(z)$ inside the unit circle that can be removed to pin down y_0 and, we find that $(1-\lambda_1\mathscr{L})(1-\lambda_2\mathscr{L})y_t=\left(y_0-\frac{\mathscr{L}w(\mathscr{L})}{a}\right)\epsilon_t$ is necessarily unstable as at most one of the two unstable autoregressive factors $(1-\lambda_k\mathscr{L})$ could be removed by a judicious choice of y_0 - that is, we have non existence of a stable solution. If, however, $|\lambda_1|,|\lambda_2|>1$, there are two singularities in $y(z)$ inside the unit circle and y_0 cannot be uniquely determined so there are multiple stable solutions - that is, we have indeterminacy. If however, $|\lambda_2|<1<|\lambda_1|$, there is one singularity in $y(z)$ inside the unit circle, namely at $z=1/\lambda_1$, and using the residue theorem[10] it can be removed to ensure the analyticity of $y(z)$ over the unit disk by setting the boundary condition on y_0 as

$$\lim_{z\to\frac{1}{\lambda_1}}(1-\lambda_1z)(1-\lambda_2z)y(z)\overset{!}{=}0=y_0-\frac{w(\lambda_1^{-1})}{\lambda_1a}\Rightarrow y_0=\frac{w(\lambda_1^{-1})}{\lambda_1a} \tag{11}$$

If w_t is white noise, i.e., $w_t=\epsilon_t$ and $w(z)=1$, then $y_0=\frac{1}{\lambda_1a}$ and the unique stable solution for the process on $y(z)$ is

$$y(z)=\frac{1}{1-\lambda_1z}\frac{1}{1-\lambda_2z}\frac{1}{a}\left(\frac{1}{\lambda_1}-z\right)=\frac{1}{1-\lambda_2z}\frac{1}{\lambda_1a}=\frac{1}{\lambda_1a}\frac{1}{1-\lambda_2z} \tag{12}$$

[10]See Ahlfors (1979, ch. 4). The Cauchy–Goursat theorem tells us the contour integral of an analytic function is zero for every closed curve in its region of convergence and Cauchy's residue theorem tells us the contour integral of function that is analytic except for isolated singularities is equal to the sum of the residues (multiplied by their winding numbers for higher order, but not essential singularities) at these singularities for a contour that contains but does not intersect these singularities. Hence, if we can choose initial conditions, such as y_0 for $y(z)$, such that these residues are zero, we then have an analytic function in the region contained by the contour.

Substituting the lag operator for z to express in the time domain gives us

$$y_t = \frac{1}{\lambda_1 a}\frac{1}{1-\lambda_2 L}\epsilon_t \Rightarrow y_t = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}\epsilon_t \tag{13}$$

For the general $w_t = w(\mathscr{L})\epsilon_t$ case,

$$y(z) = \frac{1}{1-\lambda_1 z}\frac{1}{1-\lambda_2 z}\frac{1}{a}\left(\frac{w(\lambda_1^{-1})}{\lambda_1} - zw(z)\right) = \frac{1}{1-\lambda_2 z}\frac{1}{\lambda_1 a}\frac{zw(z)-\frac{1}{\lambda_1}w(\lambda_1^{-1})}{z-\frac{1}{\lambda_1}} \tag{14}$$

Hansen and Sargent (1980) and Hansen and Sargent (1981) (see also Whiteman and Lewis (2010) for more details on this and other prediction formulas) give the equivalent to (7) for the removable singularity at $1/\lambda_1$ giving the time domain representation as

$$y_t = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}E_t\left[\frac{w(\mathscr{L})}{1-\frac{1}{\mathscr{L}\lambda_1}}\epsilon_t\right] = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}\sum_{j=0}^{\infty}\lambda_1^{-j}E_t\left[w_{t+j}\right] \tag{15}$$

Hence the requirement that one root be inside and one outside the unit circle gives the famed Blanchard and Kahn (1980) condition. Underlining the point that deriving the condition in either time or frequency domain neither alters the model itself or the associated conditions for determinacy, but simply allows us to determine unique solutions and boundary conditions of models with a different tools.

Compare this with the time domain approach. The process

$$aE_t y_{t+1} + by_t + cy_{t-1} + w_t = 0 \tag{16}$$

can be factored using λ_1 and λ_2 as

$$aE_t y_{t+1} - a(\lambda_1+\lambda_2)y_t + a\lambda_1\lambda_2 y_{t-1} + w_t = 0 \tag{17}$$

$$a(E_t y_{t+1} - \lambda_2 y_t) = a\lambda_1(y_t - \lambda_2 y_{t-1}) - w_t \tag{18}$$

define $x_t = y_t - \lambda_2 y_{t-1}$ this is

$$E_t x_{t+1} = \lambda_1 x_t - \frac{1}{a}\epsilon_t \Rightarrow x_t = \lim_{j\to\infty}\frac{1}{\lambda_1^j}E_t x_{t+j} + \frac{1}{\lambda_1 a}\sum_{j=0}^{\infty}\lambda_1^{-j}E_t\left[w_{t+j}\right] \tag{19}$$

which follows from solving forward as in Blanchard (1979). Substituting the definition of x_t gives

$$y_t = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}\sum_{j=0}^{\infty}\lambda_1^{-j}E_t\left[w_{t+j}\right] \tag{20}$$

or if w_t is white noise, i.e., $w_t = \epsilon_t$

$$y_t = \lambda_2 y_{t-1} + \frac{1}{\lambda_1 a}\epsilon_t \tag{21}$$

the same solution from the frequency domain approach above.

The question is now why bother with the frequency domain approach if it provides the same solution as the more familiar time domain approach. The answer lies in the difference in the mapping of exogenous shocks to endogenous variables. Comparing (21) to (20), the case of a general exogenous process is substantially more complicated than white noise, as we need to calculate the infinite sum $\sum_{j=0}^{\infty} \lambda_1^{-j} E_t\left[w_{t+j}\right]$. In a multivariate case with w_t being a VAR(1) process, Klein (2000) shows this can be reduced to solving a Sylvester equation and, for VARMA(p,q) processes, Meyer-Gohde and Neuhoff (2015) show this can be reduced to solving a p'th order generalized Sylvester equation and q and $p-1$ sequences of linear equations. If w_t is not a finite order ARMA process, which means $w(\mathscr{L})$ is not a rational function (i.e., cannot be expressed as $a(\mathscr{L})w(\mathscr{L}) = b(\mathscr{L})$ where $a(\mathscr{L})$ and $b(\mathscr{L})$ are finite polynomials in $\mathscr{L}$) then it is not clear how to calculate the infinite sum $\sum_{j=0}^{\infty} \lambda_1^{-j} E_t\left[w_{t+j}\right]$.

This difficulty does not, however, arise in the frequency domain case. The term analogous to the infinite forward sum in the time domain case is the frequency domain Wiener-Kolmogorov prediction formula $\frac{1}{\lambda_1 a} \frac{z w(z) - \frac{1}{\lambda_1} w(\lambda_1^{-1})}{z - \frac{1}{\lambda_1}}$. Note that instead of needing to calculate an infinite sum, we only need to be able to evaluate the function $w(z)$ on the unit circle. While an ARMA representation for w_t would permit a closed form evaluation of $w(z)$ (i.e., $w(z) = a(z)^{-1} b(z)$ with $a(z)$ and $b(z)$ known and finite polynomials, $w(z)$ is called a rational function of z), it is not necessary. In the application section, I will examine $\ln(1-\alpha z) w(z) = -z$ for $|\alpha| < 1$ from Oppenheim, Schafer, and Buck (1999, p. 117) and Oppenheim, Willsky, and Nawab (1996, p. 762). To express this in the time domain, one has to resort to the MA(∞) representation, using $\ln(1-\alpha\mathscr{L}) = -\sum_{j=1}^{\infty} \frac{\alpha^j}{j} \mathscr{L}^j$ this gives the AR(∞) representation $\sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{j+1} \mathscr{L}^j w_t = \epsilon_t$. And the z transform and digital signal processing literature, see again Oppenheim, Schafer, and Buck (1999) or Proakis and Manolakis (1996), has numerous examples of non rational transfer functions such as $w(z) = \exp(\alpha z)$, $w(z) = \sin(\alpha z)$, $w(z) = 1/\left(1 + \alpha z + \beta\sqrt{1-\gamma z}\right)$ all of which can provide parsimonious representations of complicated, from an ARMA perspective, autocorrelation patterns. But these alternatives are all equally easy to deal with as an ARMA process, we only need to be able to evaluate the function $w(z)$ for various values of z, see (4) and (5), which allow us to calculate the autocovariances and impulse responses from inverse Fourier transforms by evaluating $w(z)$ and $y(z)$ at a finite set of points z.

4. Multivariate Spectral Solution

4.1. **Problem Statement.** Consider now the general multivariate problem

$$AE_t[Y_{t+1}] + BY_t + CY_{t-1} + W_t = 0 \tag{22}$$

where Y_t is the $n_y \times 1$ vector of endogenous and W_t the $n_w \times 1$ vector of exogenous variables assumed satisfy

Assumption 4.1. *Exogenous Process* W_t
The exogenous process can be written

$$W_t = \sum_{j=0}^{\infty} \hat{W}_j \epsilon_{t-j} = W(\mathscr{L})\epsilon_t \tag{23}$$

with

$$\sum_{j=0}^{\infty} ||\hat{W}_j|| < \infty, \quad E[\epsilon_t] = 0, \quad E[\epsilon_t \epsilon_{t+i}^{'}] = 0 \;, \; \forall i \neq 0, \quad ||E[\epsilon_t \epsilon_t^{'}]|| < \infty \tag{24}$$

In his original analysis, Whiteman (1983) provided multivariate results, but assumed nonsingularity of leading coefficient matrix and distinctness of all the eigenvalues, both of which are untenable in general multivariate DSGE models.[11]

A solution of the model is a function

$$Y(z) : \mathbb{C} \to \mathbb{C}^{n_y} \tag{25}$$

that is analytic for $|z| < 1$ and solves

$$(A + zB + z^2C)Y(z) = AY(0) - zW(z) \tag{26}$$

Equation (26) follows from (22) via the Wiener-Kolmogorov prediction formula of plussing

$$\mathcal{Z}\{E_t[Y_{t+1}]\} = [\frac{Y(z)}{z}]_+ = \frac{1}{z}(Y(z) - Y(0)) \tag{27}$$

4.2. **Solvent Factorization.** I will begin by factoring the problem using Lan and Meyer-Gohde (2012) to connect the frequency domain approach with an operator approach, giving a multivariate extension of the factorization approach of the previous section from Hansen and Sargent (1980) and Hansen and Sargent (1981). I begin by formalizing the matrix quadratic equation; its solution, called a solvent; and the eigenvalues of the solvent, called latent roots of the associated lambda matrix.[12]

[11] Furthermore, both Onatski (2006) and Tan and Walker (2015) identify several inconsistencies in and correct the multivariate extension.

[12] See, e.g., Dennis, Jr., Traub, and Weber (1976, p. 835) or Gantmacher (1959, vol. I, p. 228).

Definition 4.2 (Matrix Quadratic Problem)**.** *For A, B, and $C \in \mathbb{R}^{n_y \times n_y}$, a matrix quadratic $M(X): \mathbb{C}^{n_y \times n_y} \to \mathbb{C}^{n_y \times n_y}$ is defined as*

$$M(X) \equiv AX^2 + BX + C \tag{28}$$

Definition 4.3. *Solvent of Matrix Quadratic*
$X \in \mathbb{C}^{n_y \times n_y}$ is a solvent of the matrix quadratic (28) if and only if $M(X) = 0$

Definition 4.4. *Lambda Matrix*
The lambda matrix $M(\lambda): \mathbb{C} \to \mathbb{C}^{n \times n}$ (of degree two) associated with (28) is given by

$$M(\lambda) \equiv A\lambda^2 + B\lambda + C \tag{29}$$

Its latent roots are (i) values of $\lambda \in \mathbb{C}$ such that $\det M(\lambda) = 0$ and (ii) $n_y - \operatorname{rank}\left(f_{\tilde{y}}\right)$ infinite roots.

I show that equivalents to Blanchard and Kahn's (1980) order and rank conditions are necessary and sufficient for the existence of a unique solution of Y_t adapted to the filtration. The order condition assumes a full set of latent roots with half on or inside and half outside the unit circle and the rank conditions assumes that a solution, or solvent, of (28) can be constructed with these stable roots

Assumption 4.5 (Order)**.** *There exists $2n_y$ latent roots of $A\lambda^2 + B\lambda + C$—that is, $n_y + \operatorname{rank}(A)$ finite $\lambda \in \mathbb{C} : \det(A\lambda^2 + B\lambda + C) = 0$ and $n_y - \operatorname{rank}(A)$ infinite λ—of which n_y lie inside and n_y outside the unit circle.*

Assumption 4.6 (Rank)**.** *There exists an $X \in \mathbb{R}^{n_y \times n_y}$ such that $AX^2 + BX + C = 0$ and $|eig(X)| < 1$.*

Here $eig(X)$ denotes the set of eigenvalues of X, or spectrum, $\rho(P_X)$, of the pencil $P_{I,-X}(z) \equiv I_{n_y \times n_y} z - X$ as defined in the following

Definition 4.7 (Matrix Pencil, Spectrum, and Regularity)**.** *Let $P_{FG}(z) \equiv Fz - G = 0 : \mathbb{C} \to \mathbb{C}^{n \times n}$ be a matrix-valued linear function of a complex variable; a linear matrix pencil. Its set of generalized eigenvalues or spectrum $\rho(P)$ is defined via $\rho(P) = \{z \in \mathbb{C} : \det P(z) = 0\}$. I extend the set to include infinite eigenvalues, the multiplicity of which is given by n less the rank of A. A pencil is said to be regular if $\exists z \in \mathbb{C} : \det P(z) \neq 0$.*

Theorem 4.8 (Unique Solution)**.** *Assumptions (4.5) and (4.6) are necessary and sufficient for the existence of a unique solution for Y_t to*

$$\left(A\frac{1}{\mathscr{B}}+B+C\mathscr{B}\right)E_t[Y_t]+E_t[W_t]=0 \tag{30}$$

This solution is given by

$$Y_t=-(I-X\mathscr{B})^{-1}\left(A\frac{1}{\mathscr{B}}+AX+B\right)^{-1}E_t[W_t] \tag{31}$$

$$Y_t=XY_{t-1}-\sum_{j=0}^{\infty}\left[-(AX+B)^{-1}A\right]^{-j}(AX+B)^{-1}E_t\left[W_{t+j}\right] \tag{32}$$

where X is the solvent in assumption 4.6 and $\frac{1}{\mathscr{B}}$ is Sargent's (1987, Ch. XIV) forward operator: $\frac{1}{\mathscr{B}}E_t[W_t]=E_t[W_{t+1}]$.[13]

The factorization in (31) is a multivariate extension for (potentially) singular A of Sargent's (1987, Ch. XIV), described further in Taylor (1986) and Whiteman (1983), and (32) is a multivariate version of the forward solution of Blanchard (1979).

Corollary 4.9. *The forward solution in (32) can be written as*

$$Y_t=XY_{t-1}-(AX+B)^{-1}\tilde{W}_t \tag{33}$$

where

$$\tilde{W}_t=(AX+B)^{-1}W_t+\left[-(AX+B)^{-1}A\right]=E_t\left[\tilde{W}_{t+1}\right] \tag{34}$$

and if W_t is a rational function, $W_t=P(\mathscr{L})^{-1}Q(\mathscr{L})\varepsilon_t$, where $\mathscr{L}$ is the lag operator $\mathscr{L}W_t=W_{t-1}$ and P and Q are finite matrix polynomial functions in $\mathscr{L}$ then

$$\tilde{W}_t=(AX+B)^{-1}P(\mathscr{L})^{-1}Q(\mathscr{L})\varepsilon_t+\left[-(AX+B)^{-1}A\right]E_t[w_{t+1}] \tag{35}$$

$$Q(\mathscr{B})^{-1}P(\mathscr{B})\left((AX+B)+A\frac{1}{\mathscr{B}}\right)E_t\left[\tilde{W}_t\right]=\varepsilon_t \tag{36}$$

In the frequency domain, this can be written via a $\mathscr{Z}$ transform as

$$Y(z)=-(I-Xz)^{-1}\left[\left(A\frac{1}{z}+AX+B\right)^{-1}W(z)\right]_+ \tag{37}$$

the prediction formula of plussing can be evaluated pointwise using Hansen and Sargent (1980), Hansen and Sargent (1981), and Whiteman and Lewis (2010) for the scalar u_t via

$$u_t=E_t\left[\frac{\psi(\mathscr{B})}{1-\lambda\frac{1}{\mathscr{B}}}\epsilon_t\right]=\left[\frac{\psi(\mathscr{B})}{1-\lambda\frac{1}{\mathscr{B}}}\right]_+\epsilon_t=\frac{\mathscr{L}\psi(\mathscr{L})-\lambda\psi(\lambda)}{\mathscr{L}-\lambda}\epsilon_t \tag{38}$$

[13] All proofs are in the appendix.

which translates in the frequency domain to

$$u(z) = \frac{z\psi(z) - \lambda\psi(\lambda)}{z - \lambda} \tag{39}$$

Hence to take advantage of this result, we need to diagonalize the term in (37) under the annihilation operator. Accordingly, define

$$U(z) = \left(A\frac{1}{z} + AX + B\right)^{-1} W(z) \Rightarrow \left(I - (AX+B)^{-1}(-A)\frac{1}{z}\right)U(z) = (AX+B)^{-1}W(z) \tag{40}$$

Diagonalizing $(AX+B)^{-1}(-A) = V\Lambda V^{-1}$

$$\left(I - \Lambda\frac{1}{z}\right)V^{-1}U(z) = V^{-1}(AX+B)^{-1}W(z) \tag{41}$$

and so for $\hat{U}(z) = V^{-1}U(z)$ we get

$$\left(1 - \lambda_j\frac{1}{z}\right)\hat{U}_j(z) = \left[V^{-1}(AX+B)^{-1}W(z)\right]_j \tag{42}$$

for each element j of the vector $\hat{U}(z)$ and

$$\left[\hat{U}_j(z)\right]_+ = \left[\frac{\left[V^{-1}(AX+B)^{-1}W(z)\right]_j}{1-\lambda_j\frac{1}{z}}\right]_+ = \frac{z\left[V^{-1}(AX+B)^{-1}W(z)\right]_j - \lambda_j\left[V^{-1}(AX+B)^{-1}W(\lambda_j)\right]_j}{z-\lambda_j} \tag{43}$$

And hence $[U(z)]_+$ as sought above can be constructed from

$$[U(z)]_+ = \left[V\hat{U}(z)\right]_+ = V\left[\hat{U}(z)\right]_+ \tag{44}$$

The usual restrictions and objections to diagonalizing $(AX+B)^{-1}(-A) = V\Lambda V^{-1}$ apply and one could alternatively apply a generalized Schur decomposition $\tilde{Q}^*A\tilde{Z} = \tilde{S}$ $\tilde{Q}^*(AX+B)\tilde{Z} = \tilde{T}$, where $\tilde{Q}$ and $\tilde{Z}$ are unitary, $\tilde{S}$ and $\tilde{T}$ upper triangular, and * indicates conjugate transposition. Exploiting the upper triangularity, one can begin at the last row and work recursively through $\tilde{U}(z) = \tilde{Z}^*U(z)$. Yet, the most frequent method for recovering the solvent X of the matrix quadratic problem (28) involves applying a generalized Schur decomposition and I will now show how the necessary calculations above can be recovered as a by-product.[14]

[14] Meyer-Gohde (2023) provides diagnostics for the numerical accuracy of X provided by a generalized Schur decomposition or other methods.

4.3. **Generalized Schur Decomposition/QZ Triangularization.** I will now triangularize the problem using a generalized Schur decomposition and then apply the residue theorem recursively. Rewriting (26) in first-order form (a companion linearization, see Higham and Kim (2000))

$$\Bigg(\underbrace{\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}}_{F} - z \underbrace{\begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix}}_{G}\Bigg) \underbrace{\begin{bmatrix} zY(z) \\ Y(z) \end{bmatrix}}_{\tilde{Y}(z)} = A\tilde{Y}(0) - \begin{bmatrix} 0 \\ I \end{bmatrix} zW(z) \tag{45}$$

note that $X(z)$ is analytic for $|z| < 1$ if and only if $\tilde{X}(z)$ is likewise analytic for $|z| < 1$.

Assuming away King and Watson's (1998) "mundane" source of nonuniqueness (ill-specified model, e.g., the same equation is included twice in a model)

Assumption 4.10. *The pencil $F - zG$ is regular*

I apply the complex generalized Schur decomposition to the pair (F, G).

$$Q^* F Z = S \ , \ Q^* G Z = T \tag{46}$$

where Q and Z are unitary, S and T upper triangular, and * indicates conjugate transposition.

The spectrum of the pencil $P_{DE}(z)$ is a finite set given by

$$\rho(P_{FG}) = \left\{ \begin{cases} T_{jj}/S_{jj}, & S_{jj} \neq 0 \\ \infty, & \text{otherwise} \end{cases} : j = 1, \ldots, 2n_y \right\} \tag{47}$$

where S_{jj} and T_{jj} denote the j'th row and j'th column of S and T respectively. With the continuation to infinite generalized eigenvalues,the set of generalized eigenvalues or spectrum has exactly $2n_y$ elements.[15]

Assume the model has the same number of eigenvalues inside as outside the unit circle, where eigenvalues on will be included with those in th unit circle. This is Blanchard and Kahn's (1980) order condition.

Assumption 4.11. *$\exists n_y$ eigenvalues inside the open unit circle and n_y eigenvalues outside the open unit circle*

[15]See also Klein (2000, p. 1410), Dennis, Jr., Traub, and Weber (1976, p. 835), or Golub and Van Loan (1996, p. 377), where the regularity assumption rules out $S_{jj} = T_{jj} = 0$ for some j.

Sort the decomposition with the eigenvalues in ascending absolute value and partition Q, Z, S and T according to the blocks inside and outside the unit circle. Multiplying (45) with Q^* and defining

$$\begin{bmatrix} zY(z) \\ Y(z) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} S(z) \\ U(z) \end{bmatrix} \tag{48}$$

yields

$$\left(\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} - z \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \right) \begin{bmatrix} S(z) \\ U(z) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} V(0) \\ U(0) \end{bmatrix} - Q^* \begin{bmatrix} 0 \\ I \end{bmatrix} zW(z) \tag{49}$$

Examining the lower block of the Schur decomposition

$$(S_{22} - zT_{22})U(z) = S_{22}U(0) - \tilde{W}(z) \tag{50}$$

where $\tilde{W}(z)$ is the lower n_y half of $Q^* \begin{bmatrix} 0 & I \end{bmatrix}' zW(z)$

$$\tilde{W}(z) = Q_2^* \begin{bmatrix} 0 & I \end{bmatrix}' zW(z) \tag{51}$$

Assumption 4.12. *Let the pencil*

$$S_{22} - zT_{22} \tag{52}$$

be arranged with its $M \leq n_y$ distinct eigenvalues sorted with the "infinite" eigenvalues last and the remaining eigenvalues arranged in blocks with repeated eigenvalues together and otherwise sorted in arbitrary order. For each block μ_m is defined as the reciprocal of the eigenvalue associated with the block and is the location in the complex plane of the singularities such that

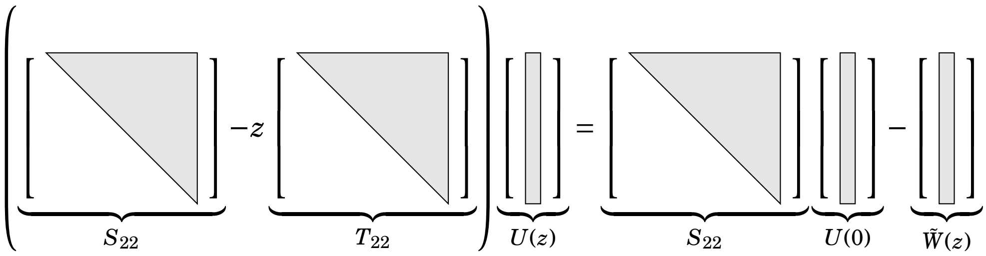
$$\det(S_{22} - zT_{22}) = 0, \ \textit{for} \ z = \mu_m, \ j = 1, \ldots, M \tag{53}$$

Hence as the eigenvalues of $P_{S_{22}T_{22}}$ are all the eigenvalues of P_{FG} outside the unit circle, the singularities μ_m, $m = 1, \ldots, M$ are all inside the unit circle.

4.4. **Simple Case: Distinct, Finite Eigenvalues.** The model has now been appropriately triangularized to be solved by the residue theorem for analytic continuation. Before I tackle the most general case, I will begin with the simple case of distinct, finite eigenvalues in $P_{S_{22}T_{22}}$.

Assumption 4.13. *The n_y eigenvalues of $P_{S_{22}T_{22}}$ are distinct, finite and outside the open unit circle.*

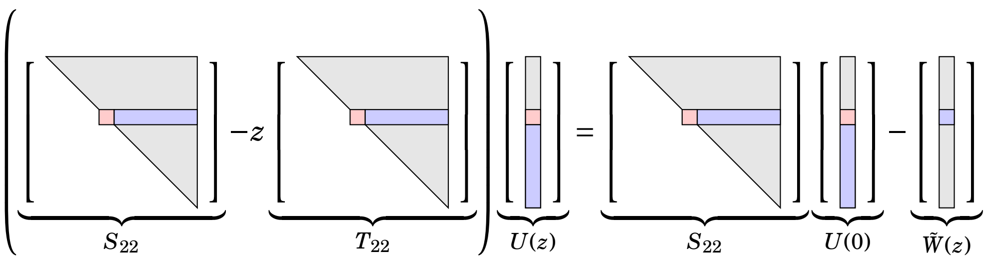
Examining the structure of the lower block of the Schur decomposition (50) in the following



(54)

reveals the triangularity that I will exploit to determine the multivariate residual. The final row is a scalar equation and, given it, the second to last row is as well. Hence, proceeding at the bottom and working up to the top will enable me to calculate the residuals individually.

Consider now the j'th row of (50) depicted as follows



(55)

and summarized in

$$\left(s_j - zt_j\right)u_j(z) + \left(S_{j+} - zT_{j+}\right)U_{j+}(z) = s_j u_j(0) + S_{j+}U_{j+}(0) - \tilde{w}_j(z) \tag{56}$$

where $u_j(z)$ is the j'th entry of $U(z)$ and $U_{j+}(z)$ are the $j+1$'th through n_y'th entries, s_j and t_j are the j'th diagonal entries of S_{22} and T_{22}—S_{22jj} and T_{22jj}, S_{j+} and T_{j+} are the j'th rows of S_{22} and T_{22} beginning at column $j+1$—$S_{22j,j+1:n_y}$ and $T_{22j,j+1:n_y}$. The equation reveals a singularity in $u_j(z)$ inside the unit circle at $\mu_j \equiv s_j/t_j$ due to $\left(s_j - zt_j\right)$. Furthermore, as the elements of $U_{j+}(z)$ solve analogous equations with singularities *not* at μ_j, $U_{j+}(\mu_j)$ is well defined and demanding the residue of $\left(s_j - zt_j\right)u_j(z)$ at μ_j be zero gives

$$\lim_{z\to\mu_j}\left(\underbrace{\left[\;\cdot\;\right]}_{s_j} - z\underbrace{\left[\;\cdot\;\right]}_{t_j}\right)\underbrace{\left[\;\cdot\;\right]}_{u_j(z)} \stackrel{!}{=} 0 \tag{57}$$

$$\Rightarrow 0 = \underbrace{\left[\;\cdot\;\right]}_{\begin{bmatrix} s_j & S_{j+}\end{bmatrix}}\underbrace{\left[\;\cdot\;\right]}_{\begin{bmatrix} u_j(0) \\ U_{j+}(0)\end{bmatrix}} - \underbrace{\left[\;\cdot\;\right]}_{\tilde{w}_j(\mu_j)} - \left(\underbrace{\left[\;\cdot\;\right]}_{S_{j+}} - \mu_j\underbrace{\left[\;\cdot\;\right]}_{T_{j+}}\right)\underbrace{\left[\;\cdot\;\right]}_{U_{j+}(\mu_j)} \tag{58}$$

or summarized

$$\lim_{z\to\mu_j}\left(s_j - zt_j\right)u_j(z) \stackrel{!}{=} 0 = -\left(S_{j+} - \mu_j T_{j+}\right)U_{j+}(\mu_j) + s_j u_j(0) + S_{j+}U_{j+}(0) - \tilde{w}_j(\mu_j) \tag{59}$$

which can be solved for $u_j(0)$ (note the assumption of finite eigenvalues ensures that s_j is non-zero, this assumption will be relaxed below) taking $U_{j+}(0)$ as having been solved previously (these are the elements of $U(0)$ after $u_j(0)$ and I proceed recursively starting at the last entry) and $U_{j+}(\mu_j)$ as recoverable from

$$\left(\underbrace{\left[\;\cdot\;\right]}_{S_{22\,j+1:n_y,j+1:n_y}} - \mu_j\underbrace{\left[\;\cdot\;\right]}_{T_{22\,j+1:n_y,j+1:n_y}}\right)\underbrace{\left[\;\cdot\;\right]}_{U_{j+}(\mu_j)} = \underbrace{\left[\;\cdot\;\right]}_{S_{22\,j+1:n_y,j+1:n_y}}\underbrace{\left[\;\cdot\;\right]}_{U_{j+}(0)} - \underbrace{\left[\;\cdot\;\right]}_{\tilde{W}_{j+}(\mu_j)} \tag{60}$$

or

$$\left(S_{22\,j+1:n_y,j+1:n_y} - \mu_j T_{22\,j+1:n_y,j+1:n_y}\right)U_{j+}(\mu_j) = S_{22\,j+1:n_y,j+1:n_y}U_{j+}(0) - \tilde{W}_{j+}(\mu_j) \tag{61}$$

which can be solved for $U_{j+}(\mu_j)$ as $S_{22\,j+1:n_y,j+1:n_y} - \mu_j T_{22\,j+1:n_y,j+1:n_y}$ is non singular as I assumed distinct eigenvalues and μ_j is singularity associated with the j'th diagonal elements of S_{22} and T_{22}.

Hence the triangularization of the generalized Schur decomposition will allow me to apply the residue approach row by row proceeding recursively from the last row to the first. If there are n_y eigenvalues outside the unit circle (singularities of $U(z)$ on the unit disc), there are as many residue restrictions as initial conditions $U(0)$ (this is Blanchard and Kahn's (1980) order condition) and if Z_{22}^* is full rank, $U(z)$ and $U(0)$ can be mapped into $Y(z)$ and $Y(0)$ uniquely (this is Blanchard and Kahn's (1980) rank condition), as I summarize in the following

Theorem 4.14 (Solution of $Y(z)$ with Distinct, Finite Unstable Eigenvalues)**.** *Let assumption 4.13 hold and let Z_{22}^* be of full rank, then*

$$Y(z) = -\left(Z_{22}^*\right)^{-1} Z_{21}^* z Y(z) + \left(Z_{22}^*\right)^{-1} U(z) \quad \forall z \in \{z \in \mathbb{C} \mid |z| \leq 1, z \neq \mu_m \text{ for } m = 1,2,\ldots,M\} \tag{62}$$

where

$$U(z) = (S_{22} - zT_{22})^{-1} \left(S_{22}U(0) - \tilde{W}(z)\right) \tag{63}$$

for a W_t that satisfies (4.1) with $\tilde{W}_t$ related to W_t as in(51).

The elements of $U(0) = \begin{bmatrix} \tilde{u}_1(0) & \ldots & \tilde{u}_{n_y}(0) \end{bmatrix}'$ are given recursively, starting with $j = n_y$, by

$$U(z) = (S_{22} - zT_{22})^{-1} \left(S_{22}U(0) - \tilde{W}(z)\right) \tag{64}$$

the elements of $U(0) = \begin{bmatrix} \tilde{u}_1(0) & \ldots & \tilde{u}_{n_y}(0) \end{bmatrix}'$ are given recursively, starting with $j = n_y$, by

$$u_j(0) = s_j^{-1} \left[\left(S_{j+} - \mu_j T_{j+}\right) \left(S_{j+}^+ - \mu_j T_{j+}^+\right)^{-1} \left(S_{j+}^+ U_{j+}(0) - \tilde{W}_{j+}(\mu_j)\right) - S_{j+}U_{j+}(0) - \tilde{w}_j(\mu_j) \right] \tag{65}$$

where $\tilde{W}_{j+}(\mu_j) = \begin{bmatrix} \tilde{w}_{j+1}(\mu_j) & \ldots & \tilde{w}_{n_y}(\mu_j) \end{bmatrix}'$, $U_{j+}(0)$, s_j, S_{j+} and T_{j+} are as defined following (56) and S_{j+}^+ and T_{j+}^+ are the remaining lower right elements of S_{22} and T_{22} beginning at row and column $j+1$—$S_{22_{j+1:n_y,j+1:n_y}}$ and $T_{22_{j+1:n_y,j+1:n_y}}$.

Note that the theorem gives the value for $Y(z)$ everywhere on the unit disk except for at the singularities. This will generally be sufficient as, e.g., the impulse responses and autocovariances can be determined be evaluating $Y(z)$ along the unit circle.[16] The foregoing also restricted the analysis to the case of finite eigenvalues, ensuring s_j is non zero for all j, and distinct eigenvalues, which allowed the row by row application of the residue theorem. Consider briefly the consequences of an infinite eigenvalue, which must be last from the sorting above, the n_y'th row of (50) is

$$\left(s_{n_y} - zt_{n_y}\right) u_{n_y}(z) = s_{n_y} u_{n_y}(0) - \tilde{w}_{n_y}(z) \tag{66}$$

[16] If one actually needs the value of $Y(z)$ at one of the singularities, as might be the case with information rigidites, see Meyer-Gohde and Tzaawa-Krenzler (2023), the appendix contains the formulas.

and for $\mu_{n_y} = 0$, it must be the case that $s_{n_y} = 0$ - see also (47) - so the foregoing becomes

$$z t_{n_y} u_{n_y}(z) = \tilde{w}_{n_y}(z) \Leftrightarrow u_{n_y}(z) = t_{n_y}^{-1} \frac{\tilde{w}_{n_y}(z) - \tilde{w}_{n_y}(0)}{z} \tag{67}$$

as $\tilde{w}_{n_y}(0) = 0$. Taking the limit as $z \to 0$ gives

$$u_{n_y}(0) = t_{n_y}^{-1} \tilde{w}_{n_y}^{(1)}(0) \tag{68}$$

where $\tilde{w}_{n_y}^{(1)}(0)$ is the derivative of $\tilde{w}_{n_y}(z)$ with respect to z evaluated at $z = 0$. Hence, for the infinite eigenvalue, the residue theorem is not applied to recover $u_{n_y}(0)$ as it coincides with $u_{n_y}(\mu_{n_y})$, but instead requires differentiation.

Likewise I assumed distinct eigenvalues. Consider briefly the consequences of repeated eigenvalues and assume for ease of exposition that they are the last two. The n_y'th row of (50) is

$$\left(s_{n_y} - z t_{n_y}\right) u_{n_y}(z) = s_{n_y} u_{n_y}(0) - \tilde{w}_{n_y}(z) \tag{69}$$

and the associated residue is

$$\lim_{z \to \mu_{n_y}} \left(s_{n_y} - z t_{n_y}\right) u_{n_y}(z) \stackrel{!}{=} 0 = s_{n_y} u_{n_y}(0) - \tilde{w}_{n_y}(\mu_{n_y}) \tag{70}$$

so $u_{n_y}(0) = \tilde{w}_{n_y}(\mu_{n_y})/s_{n_y}$ and $\mu_{n_y} = s_{n_y}/t_{n_y}$. The $n_y - 1$'th row of (50) is

$$\left(s_{n_y-1} - z t_{n_y-1}\right) u_{n_y-1}(z) + \left(s_{n_y-1+} - z s_{n_y-1+}\right) u_{n_y}(z) = s_{n_y-1} u_{n_y-1}(0) + s_{n_y-1+} u_{n_y}(0) - \tilde{w}_{n_y-1}(z) \tag{71}$$

and the associated residue is

$$\lim_{z \to \mu_{n_y-1}} \left(s_{n_y-1} - z t_{n_y-1}\right) u_{n_y-1}(z) \stackrel{!}{=} 0 \tag{72}$$

$$\Rightarrow 0 = s_{n_y-1} u_{n_y-1}(0) + s_{n_y-1+} u_{n_y}(0) - \tilde{w}_{n_y-1}(\mu_{n_y-1}) - \left(s_{n_y-1+} - \mu_{n_y-1} s_{n_y-1+}\right) u_{n_y}(\mu_{n_y-1}) \tag{73}$$

where now two terms from the previous (lower) row, $u_{n_y}(0)$ and $u_{n_y}(\mu_{n_y-1})$ are needed to be able to solve for $u_{n_y-1}(0)$. Now $u_{n_y}(0)$ is not an issue as it was solved for in (70). The term $u_{n_y}(\mu_{n_y-1})$ can be recovered from

$$u_{n_y}(z) = \left(s_{n_y} - z t_{n_y}\right)^{-1} \left(s_{n_y} u_{n_y}(0) - \tilde{w}_{n_y}(z)\right) \tag{74}$$

for every $z \neq \mu_{n_y}$. But that is of course the value we need when eigenvalues repeat $\mu_{n_y-1} = \mu_{n_y}$. In this case, the equation above is undefined $u_{n_y}(z) = 0^{-1}0$ and L'Hôpital gives

$$u_{n_y}(\mu_{n_y}) = \left(\mu_{n_y} t_{n_y}\right)^{-1} \left(\tilde{w}_{n_y}^{(1)}(\mu_{n_y})\right) = s_{n_y}^{-1} \tilde{w}_{n_y}^{(1)}(\mu_{n_y} \tag{75}$$

which is equal to the term $u_{n_y}(\mu_{n_y-1})$ sought above when $\mu_{n_y-1} = \mu_{n_y}$. As above with infinite eigenvalues, the derivative of $\tilde{w}_{n_y}(z)$ with respect to z is needed.

I now turn to the general case with repeated and infinite eigenvalues where, as is familiar from say the Jordan decomposition, derivatives will be involved - the mapping however will only require the explicit differentiation of the exogenous process $\tilde{W}(z)$.

4.5. General Case: Potentially Repeated, Potentially Infinite Eigenvalues.

Consider now the case of n_y eigenvalues outside the unit circle, potentially infinite (or rather with associated zero values of s_j, see above), and potentially repeated so that there are $M \leq n_y$ distinct eigenvalues. As stated above, let the pencil

$$S_{22} - zT_{22} \tag{76}$$

be arranged with its $M \leq n_y$ distinct eigenvalues sorted with the "infinite" eigenvalues last and the remaining eigenvalues arranged in blocks with repeated eigenvalues together and otherwise sorted in arbitrary order. The algorithm closely resembles Higham's (2008, Ch. 9) "Schur–Parlett Algorithm" but extended here to work on a pencil with the generalized Schur decomposition instead of the matrix function approach based on the standard Schur decomposition.

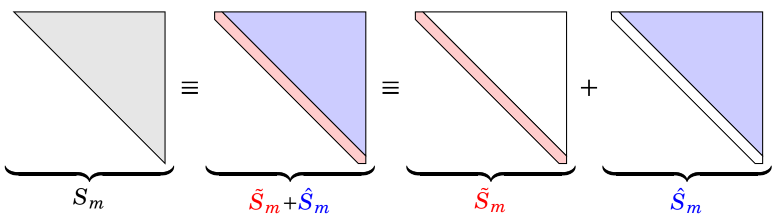
Consider block m of dimension k_m of (50), the potentially repeated or infinite eigenvalue analogue to (56),

$$(S_m - zT_m)U_m(z) + (S_{m+} - zT_{m+})U_{m+}(z) = S_m U_m(0) + S_{m+}U_{m+}(0) - \tilde{W}_m(z) \tag{77}$$

Note that any singularities at zero, the infinite eigenvalues, must be contained in the last block M. Hence, for block $m < M$, the singularity μ_m satisfies

$$\tilde{S}_m - \underset{1\times 1}{\mu_m}\tilde{T}_m = \underset{k_m\times k_m}{0} \tag{78}$$

where $\tilde{S}_m$ is the diagonal of S_m and $\tilde{T}_m$ of S_m. That is, the singularity of this block is μ_m and is given by $\mu_m = S_{mjj}/T_{mjj}$ for any and all $j = 1,\ldots,k_m$. Define $\hat{S}_m = S_m - \tilde{S}_m$ and $\hat{T}_m = T_m - \tilde{T}_m$ as the strictly upper triangular matrices that contain the off diagonal elements of S and T.



(79)

The matrices $\hat{S}_m$ and $\hat{T}_m$ are strictly diagonal, containing nonzero elements at most on the superdiagonal and above. Accordingly, they are nilpontent of degree k_m. That is,

$$\hat{S}_m^{k_m} = \hat{T}_m^{k_m} = \underset{k_m \times k_m}{0} \tag{80}$$

As is generally the case with functions of matrices, see Higham (2008, ch. 1) and Gantmacher (1959, ch. 5), we will be evaluating the matrix function $U(z)$ on the spectrum which requires us to be able to construct the derivatives of $U(z)$ with respect to z to the order that corresponds to one less than the index of the repeated eigenvalue. See the conclusion of the previous subsection two repeated roots required the first derivative to be calculated. Accordingly

Proposition 4.15 (Derivative of block m)**.** *The n'th derivative of (77) with respect to z is*

$$(S_m - zT_m)U_m^{(n)}(z) - nT_m U_m^{(n-1)}(z) + (S_{m+} - zT_{m+})U_{m+}^{(n)}(z) - nT_{m+}U_{m+}^{(n-1)}(z) = -\tilde{W}_m^{(n)}(z) \tag{81}$$

As long as $z \neq \mu_m$, this gives a recursive expression for $U_m^{(n)}(z)$ given by

Corollary 4.16 (Recursive function and derivative of block m at $z \neq \mu_m$)**.** *$U_m(z)$ at $z \neq \mu_m$ is given by*

$$U_m(z) = (S_m - zT_m)^{-1}\left(S_m U_m(0) + S_{m+}U_{m+}(0) - \tilde{W}_m(z) - (S_{m+} - zT_{m+})U_{m+}(z)\right) \tag{82}$$

and its n'th derivative with respect to z at $z \neq \mu_m$

$$U_m^{(n)}(z) = (S_m - zT_m)^{-1}\left(nT_m U_m^{(n-1)}(z) + nT_{m+}U_{m+}^{(n-1)}(z) - \tilde{W}_m^{(n)}(z) - (S_{m+} - zT_{m+})U_{m+}^{(n)}(z)\right) \tag{83}$$

Note that corollary 4.16 allows us to determine the value of $U_m(z)$, the current block, and its z derivative at every value of z on the unit disk apart from $z = mu_m$, taking $U_m(0)$, the value $U_{m+}(0)$ and the functions from previous (that is, higher) blocks $U_{m+}(z)$ and their derivatives $U_{m+}^{(n)}(z)$ as well as of the exogenous process $\tilde{W}_m(z)$ and its derivatives $\tilde{W}_m^{(n)}(z)$ as given. Hence, if we proceed recursively through the blocks, we still need to determine $U_m(0)$ and then we can proceed to the next block as the grouping of eigenvalues ensures that the block m contains all equations associated with singularities equal to μ_m. [17]

[17] Technically, I also need to provide values for $U_m(z)$ and its z derivative at $z = mu_m$ to be able to calculate $U_m(z)$ for every value on the unit disk. Practically, this isn't necessary as the calculations will be done using discrete approximations with inverse Fourier transforms. But, as we'll see below, $U_m(0)$ will be uncovered by considering $U_m(z)$ at $z = mu_m$ and $U_m(z)$ and its z derivative at $z = mu_m$ will follow as by-products.

To approach calculating the (potentially) multivariate block $U_m(z)$ at its singularity, note that decomposing matrices into a diagonal matrix containing the eigenvalues and a nilpotent matrix containing the remaining elements is a defining feature of the Jordan decomposition, see Gantmacher (1959, ch. 7) and Horn and Johnson (2013, Ch. 3.2.7), and for our matrix function approach, of an atomic block following Higham (2008, Ch. 9). This nilpotency enables me to formulate a finite recursion in the derivatives of a block m at its singularity μ_m as follows

Corollary 4.17 (Recursive derivative of block m at $z = \mu_m$)**.** *The n'th derivative of (77) with respect to z at $z = \mu_m$, (81) evaluated at $z = \mu_m$, is*

$$\tilde{U}_m^{(0)}(\mu_m) = \sum_{j=1}^{k_m-1} \frac{1}{j!}\Theta_m^j \left[\tilde{W}_m^{(j)}(\mu_m) + (S_{m^+}\mu_m T_{m^+})U_{m^+}^{(j)}(\mu_m) - jT_{m^+}U_{m^+}^{(j-1)}(\mu_m)\right] \tag{84}$$

where $\Theta_m \equiv \left(\hat{S}_m - \mu_m \hat{T}_m\right) T_m^{-1}$ *and* $\Theta_m^{k_m}$ *is nilpotent with* $\Theta_m^{k_m} = 0$

I am now in a position to set the residue of (77) to zero and recover the vector $U_m(0)$, the initial conditions associated with block m

Proposition 4.18 (Residue of block m)**.** *Demanding the residue of (77) be zero requires*

$$U_m(0) = S_m^{-1}\left[\tilde{W}_m(\mu) + (S_{m^+} - \mu_m T_{m^+})U_{m^+}(\mu_m) - S_{m^+}U_{m^+}(0)\right] + S_m^{-1}\tilde{U}_m(\mu_m) \tag{85}$$

and (A-25) or, expressed together,

$$U_m(0) = S_m^{-1}\left[\tilde{W}_m(\mu) + (S_{m^+} - \mu_m T_{m^+})U_{m^+}(\mu_m) - S_{m^+}U_{m^+}(0)\right] \tag{86}$$

$$+ S_m^{-1}\sum_{j=1}^{k_m-1} \frac{1}{j!}\Theta_m^j \left[\tilde{W}_m^{(j)}(\mu_m) + (S_{m^+}\mu_m T_{m^+})U_{m^+}^{(j)}(\mu_m) - jT_{m^+}U_{m^+}^{(j-1)}(\mu_m)\right] \tag{87}$$

As I will explicitly allow for singularities at zero, which will be sorted to the final block, consider finally this final, M'th block of dimension k_M of (50),

$$(S_M - zT_M)U_M(z) = S_M U_M(0) - \tilde{W}_M(z) \tag{88}$$

if $\mu_M \neq 0$ then I can proceed exactly as above, setting the residue at μ_M to zero. Otherwise, $\mu_M = 0$ which implies the diagonal elements of S_M are zero and $\tilde{S}_M = \underset{k_M \times k_M}{0}$. Hence, $\hat{S}_M = S_M$ and $\hat{T}_M = T_M - \tilde{T}_M$. Using this, (88) can be expressed as

$$U_M(z) = T_M^{-1}\hat{S}_M \frac{U_M(z) - U_M(0)}{z} + T_M^{-1}\frac{\tilde{W}_M(z)}{z} = T_M^{-1}\hat{S}_M \frac{U_M(z) - U_M(0)}{z} + T_M^{-1}\frac{\tilde{W}_M(z) - \tilde{W}_M(0)}{z} \tag{89}$$

as $\tilde{W}_M(0)$. Letting $z \to \mu_M = 0$

$$U_M(0) = T_M^{-1}\hat{S}_M U_M^{(1)}(0) + T_M^{-1}\tilde{W}_M^{(1)}(0) \tag{90}$$

Note that this is implied by corollary 4.17 with $\mu_M = 0$ and hence

Proposition 4.19 (Block M - Singularity at 0)**.** *If $\mu_M = 0$, then the coefficients $U_M(0)$ satisfy*

$$U_M(0) = T_M^{-1}\sum_{j=0}^{k_M-1}\frac{1}{(j+1)!}\Theta_M^j\tilde{W}_M^{(j+1)}(0) \tag{91}$$

where $\Theta_M \equiv \hat{S}_M T_M^{-1}$

I collect the individual results from above together to provide the following theorem

Theorem 4.20 (Frequency Domain Solution of $Y(z)$ for Multivariate Linear Models)**.** *Let assumption 4.11 hold and let Z_{22}^* be of full rank, then*

$$Y(z) = -\left(Z_{22}^*\right)^{-1}Z_{21}^* zY(z) + \left(Z_{22}^*\right)^{-1}U(z) \quad \forall z \in \{z \in \mathbb{C} \mid |z| \le 1, z \ne \mu_m \text{ for } m = 1,2,\dots,M\} \tag{92}$$

where

$$U(z) = (S_{22} - zT_{22})^{-1}\left(S_{22}U(0) - \tilde{W}(z)\right) \tag{93}$$

for a W_t that satisfies (4.1) with $\tilde{W}_t$ related to W_t as in (51).

The pencil $S_{22} - zT_{22}$ is sorted into M blocks in accordance with assumption 4.12 and S, T, Q and Z follow from the generalized Schur decomposition (46). The elements of $U(0) = \begin{bmatrix}\tilde{u}_1(0) & \dots & \tilde{u}_M(0)\end{bmatrix}'$ are given recursively blockwise, starting with $m = M$, by

$$U_M(0) = T_M^{-1}\sum_{j=0}^{k_M-1}\frac{1}{(j+1)!}\Theta_M^j\tilde{W}_M^{(j+1)}(0) \tag{94}$$

with $\Theta_M \equiv \hat{S}_M T_M^{-1}$ if $\mu_M = 0$ and otherwise

$$U_m(0) = S_m^{-1}\left[\tilde{W}_m(\mu) + (S_{m^+} - \mu_m T_{m^+})U_{m^+}(\mu_m) - S_{m^+}U_{m^+}(0)\right] \tag{95}$$

$$+ S_m^{-1}\sum_{j=1}^{k_m-1}\frac{1}{j!}\Theta_m^j\left[\tilde{W}_m^{(j)}(\mu_m) + (S_{m^+}\mu_m T_{m^+})U_{m^+}^{(j)}(\mu_m) - jT_{m^+}U_{m^+}^{(j-1)}(\mu_m)\right] \tag{96}$$

where $\Theta_m \equiv \left(\hat{S}_m - \mu_m\hat{T}_m\right)T_m^{-1}$ and

$$U_{m+}(\mu_m) = \left(S_{m+}^+ - \mu_m T_{m+}^+\right)^{-1}\left(S_{m+}^+U_{m+}(0) - \tilde{W}_{m+}(\mu_m)\right) \tag{97}$$

where $\tilde{W}_{m+}(\mu_j) = \begin{bmatrix} \tilde{w}_{m+1}(\mu_j) & \dots & \tilde{w}_M(\mu_j) \end{bmatrix}'$, $U_{m+}(0)$, S_m, T_m, S_{m+} *and* T_{m+} *are as defined following (78) and* S^+_{j+} *and* T^+_{j+} *are the remaining lower right elements of* S_{22} *and* T_{22} *beginning after the rows and columns of block m.*

The existence and uniqueness properties are summarized in the dimensions of (48) or rather the inverse mapping

$$\begin{bmatrix} Z^*_{11} & Z^*_{12} \\ Z^*_{21} & Z^*_{22} \end{bmatrix} \begin{bmatrix} zY(z) \\ Y(z) \end{bmatrix} = \begin{bmatrix} S(z) \\ U(z) \end{bmatrix} \tag{98}$$

examining this at $z = 0$ gives

$$\begin{bmatrix} Z^*_{11} & Z^*_{12} \\ Z^*_{21} & Z^*_{22} \end{bmatrix} \begin{bmatrix} \underset{n_y \times 1}{0} \\ Y(0) \end{bmatrix} = \begin{bmatrix} S(0) \\ U(0) \end{bmatrix} \tag{99}$$

and the last block equation is

$$\underset{n_u \times n_y}{Z^*_{22}} Y(0) = U(0) \tag{100}$$

Apparently, my ability to recover $Y(0)$ from $U(0)$ hinges on the dimensions and rank of Z^*_{22}. This is the equivalent to Blanchard and Kahn's (1980) order and rank conditions. I summarize this in the following

Corollary 4.21 (Existence and Uniqueness of the Solution in Theorem 4.20)**.** *There exists a unique, stable solution for* $Y(z)$ *given a unique stable* $U(z)$ *if and only if*

(1) The dimensions of $U(z)$ *and* $Y(z)$ *coincide* ($n_u = n_y$) *and*
(2) the matrix Z^*_{22} *is of full rank* ($\text{rank}(Z^*_{22}) = n_y = n_u$)

If $n_u < n_y$ *there are not enough elements in* $U(0)$ *to determine all the elements of* $Y(0)$ *uniquely - an analytic* $Y(z)$ *on the unit disc is non-unique or indeterminate. If* $n_u > n_y$ *there are more elements in* $U(0)$ *than can be associated with elements of* $Y(0)$ *- an analytic* $Y(z)$ *on the unit disc does not exist.*

5. Factorized Solution Redux

The problem expressed above in terms of the backshift, $\mathscr{B}$, and forward operator, $\frac{1}{\mathscr{B}}$, in (30), can be factored with a solvent X as

$$\left(A\frac{1}{\mathscr{B}} + AX + B\right)(I - X\mathscr{B})E_t[Y_t] = -E_t[W_t] = 0 \tag{101}$$

Given the QZ decomposition above, see (46), sorted with the eigenvalues in ascending absolute value and Q, Z, S and T partitioned according to the blocks inside and outside the unit circle, this can be diagonalized as in the following.

Corollary 5.1 (Diagonalized Factor Solution). *Let assumptions (4.5) and (4.6) hold. The solution to (30) in theorem 4.8*

$$\left(A\frac{1}{\mathscr{B}}+B+C\mathscr{B}\right)E_t[Y_t]+E_t[W_t]=0 \tag{102}$$

can be expressed in terms of the QZ decomposition above, see (46), sorted with the eigenvalues in ascending absolute value and Q, Z, S and T partitioned according to the blocks inside and outside the unit circle, following assumption 4.12, as

$$Y_t=-Z_{11}\left(S_{11}-T_{11}\mathscr{B}\right)^{-1}Z_{11}^{-1}R\left(S_{22}\frac{1}{\mathscr{B}}-T_{22}\right)^{-1}Q_{22}^{*}E_t[W_t] \tag{103}$$

where $R\equiv Z_{11}Q_{11}^{-1}Z_{22}-Z_{12}Q_{11}^{-1}Z_{12}$ or, equivalently,

$$Y_t=-\left(Z_{21}^{*}\mathscr{B}+Z_{22}^{*}\right)^{-1}\left(S_{22}\frac{1}{\mathscr{B}}-T_{22}\right)^{-1}Q_{22}^{*}E_t[W_t] \tag{104}$$

Inspection shows that the final representation corresponds one-to-one via a $\mathscr{Z}$ transform to theorem 4.20, which in its factored form is

$$Y_t=-\left(Z_{22}^{*}\right)^{-1}Z_{21}^{*}Y_{t-1}-\left(Z_{22}^{*}\right)^{-1}\left(S_{22}\frac{1}{\mathscr{B}}-T_{22}\right)^{-1}Q_{22}^{*}E_t[W_t] \tag{105}$$

Giving the forward solution

$$Y_t=-\left(Z_{22}^{*}\right)^{-1}Z_{21}^{*}Y_{t-1}+\left(T_{22}Z_{22}^{*}\right)^{-1}\sum_{j=0}^{\infty}\left[(T_{22})^{-1}S_{22}\right]^{-j}Q_{22}^{*}E_t\left[W_{t+j}\right] \tag{106}$$

or in the z-domain

$$Y(z)=-\left(Z_{21}^{*}z+Z_{22}^{*}\right)^{-1}\left[\left(S_{22}\frac{1}{z}-T_{22}\right)^{-1}Q_{22}^{*}W(z)\right]_{+} \tag{107}$$

and the recursive triangular resolution described at the end of section 4.2 to enable the pointwise evaluation of plussing using the prediction formula of Hansen and Sargent (1980), Hansen and Sargent (1981), and Whiteman and Lewis (2010) is exactly what section 4.5 provides.

6. MOMENTS AND IMPULSE RESPONSE ANALYSIS

Having solved for a frequency domain solution $Y(z)$ from the previous section, the next step is to analyze it. In particular, I will show how to recover the impulse responses of the model and its autocovariance function. The latter can then be used to evaluate the likelihood function under Gaussianity.

Given $Y(z)$ we can write the time domain representation vie the inverse z-transform as

$$Y_t = Y(\mathscr{L})\epsilon_t = \sum_{j=0}^{\infty} \hat{Y}_j \epsilon_{t-j} \tag{108}$$

the coefficients $\hat{Y}_j$ are the vector $MA(\infty)$ coefficients and, hence, contain the impulse responses: $\hat{Y}_{j_{k,l}}$ is the response of endogenous variable k - the k'th element of Y_t - to a shock in exogenous variable l - the l'th element of ϵ_t - j periods ago.

Following, e.g., Sargent (1987, ch. XI) the converse of Riesz-Fischer Theorem gives an equivalence (a one-to-one and onto transformation) between the space of analytic functions in unit disk $y(z)$ and the space of squared summable sequences $\sum_{j=0}^{\infty} y_j^2 < \infty$, corresponding to the inverse z-transform of the sequence, $y(z) = \sum_{j=0}^{\infty} y_j z^j$. The inversion formula from section 2 is

$$\hat{Y}_j = \frac{1}{2\pi}\int_{\Gamma} Y(z) z^{-j} dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{-i\omega}) e^{ji\omega} d\omega \tag{109}$$

Numerically, the coefficients $\hat{Y}_j$ can be recovered via in inverse fast Fourier transformation.

I review how to evaluating the likelihood by calculating the sequence of autocovariances spectrally, see, e.g., Meyer-Gohde and Neuhoff (2015) for more details. A linear combination of elements of Y_t, e.g., observables, possess the MA(∞) representation

$$X_t = \underset{n_y \times n_x}{\Upsilon^X}, \; X_t = \Upsilon^X \left(\underset{n_x \times n_x}{I} - \Lambda L\right)^{-1} \left[\Phi(L) P(L)^{-1} Q(L) + \Theta(L)\right] \epsilon_t \tag{110}$$

E.g., Sargent (1987) or Uhlig (1999) show the autocovariances of Y_t, $\Gamma_n \doteq E\left[Y_t Y_{t-n}'\right]$, are

$$\Gamma_n = \int_{-\pi}^{\pi} G(\omega) e^{i\omega n} d\omega \tag{111}$$

the inverse Fourier transformation of the spectral density of Y_t, $G(\omega)$ given by

$$G(\omega) \doteq H(-\omega)\Sigma H(\omega)', \; H(\omega) = \Upsilon^X \left(\underset{n_x \times n_x}{I} - \Lambda e^{i\omega}\right)^{-1} \left[\Phi\left(e^{i\omega}\right) P\left(e^{i\omega}\right)^{-1} Q\left(e^{i\omega}\right) + \Theta\left(e^{i\omega}\right)\right] \tag{112}$$

T observations of Y_t, $Y = [Y_1'Y_2' \dots Y_T']'$, are then normal with block Toeplitz covariance matrix

$$\Psi = \begin{bmatrix} \Gamma_0 & \Gamma_1' & \dots & \Gamma_{T-1}' \\ \Gamma_1 & \Gamma_0 & \dots & \Gamma_{T-2}' \\ \vdots & & \ddots & \vdots \\ \Gamma_{T-1} & & \dots & \Gamma_0 \end{bmatrix} \tag{113}$$

with autocovariances, Γ_n, from (111), the log-likelihood of parameters ς given the data is

$$\mathscr{L}(\varsigma|Y) = -0.5pTln(2\pi) - 0.5ln(det(\Psi(\vartheta))) - 0.5Y'\Psi(\vartheta)^{-1}Y \tag{114}$$

$ln(det(\Psi(\vartheta)))$ and $Y'\Psi(\vartheta)^{-1}Y$ can be calculated with (113) following, e.g., the block Levinson algorithm in Meyer-Gohde (2010).

7. Nonrecursive Transfer Functions in a New Keynesian model

To illustrate the usefulness of the method introduced above, I will estimate and analyze the basic New Keynesian model laid out in Herbst and Schorfheide (2015). This is a loglinearized textbook version of the model of nominal rigidities at the cornerstone of many macroeconomic policy analyses such as Smets and Wouters (2007). In contrast to Herbst and Schorfheide (2015) who assume AR(1) processes for technology and government expenditures, I will also examine MA(1) specifications - which can of course easily be examined with existing methods - and two processes from the control literature that feature nonlinear functions of the lag operator to capture their autocorrelation - which cannot be analyzed with time domain methods and whose estimation is beyond the reach of other existing methods. I limit all four specifications to a single parameter, putting them on equal footing in terms of parsimony, and find that the specification I term log harmonic lag is referred by a Bayesian analysis, owing to its hump-shaped autocorrelation and impulse pattern - a common feature of macroeconomic time series, Cogley and Nason (1995).

Introducing the model, I start with the New Keynesian Phillips curve that summarizes the supply side of the economy

$$\hat{\pi}_t = \beta\mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t) \tag{115}$$

where $\hat{\pi}_t$ is inflation, $\hat{y}_t$ output and $\hat{g}_t$ as flex price output - hence $\hat{y}_t - \hat{g}_t$ is the output gap. The parameter β is the representative household's subjective discount factor and $\kappa = \tau\frac{1-\nu}{\nu\pi^2\phi}$ captures the short run tradeoff int he Phillips curve with ϕ being the measure of price rigidity via quadratic adjustment costs, π steady state inflation, $1/\nu$ is the elasticity of substitution between differentiated goods, and $1/\tau$ is the representative household's elasticity of intertemporal substitution. Now turning to the demand side, given by a dynamic IS equation,

$$\hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] - \frac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \mathbb{E}_t[\hat{z}_{t+1}]\right) + \hat{g}_t - \mathbb{E}_t[\hat{g}_{t+1}] \tag{116}$$

$\hat{z}_t$ is fluctuations in aggregate productivity growth and $\hat{R}_t$ is the nominal interest rate controlled by the central bank via the Taylor rule

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + \left(1-\rho_R\right)\psi_1 \hat{\pi}_t + \left(1-\rho_R\right)\psi_2(\hat{y}_t - \hat{g}_t) + \epsilon_{R,t} \tag{117}$$

where $\epsilon_{R,t}$ is a monetary policy shock, ρ_R is the degree of interest rate smoothing, ψ_1 the response of monetary policy to inflation and ψ_2 to the output gap.

Herbst and Schorfheide (2015), following standard practice, choose AR(1) specifications for government expenditures

$$\hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_{g,t} \tag{118}$$

and the technology process or aggregate productivity growth fluctuations[18]

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z,t} \tag{119}$$

The AR(1) assumption is an appealing mechanism for generating persistence in DSGE models with a long tradition, methodologically Blanchard and Kahn (1980), theoretically Kydland and Prescott (1982) and empirically Nelson and Plosser (1982). Yet this is potentially an oversimplification and I will consider four separate specifications for the exogenous processes $\hat{z}_t$ and $\hat{g}_t$. Alongside the original AR(1)

$$\hat{m}_t^{AR} = \frac{1}{1-\rho_m L}\epsilon_{m,t}, \;\; m \in \{g,z\} \tag{120}$$

and MA(1) specification

$$\hat{m}_t^{MA} = \left(1+\rho_m L\right)\epsilon_{m,t}, \;\; m \in \{g,z\} \tag{121}$$

a log lag operator specification - this is from Oppenheim, Schafer, and Buck (1999, p. 117) and Oppenheim, Willsky, and Nawab (1996, p. 762), but modified to have a unit coefficient

[18] Aggregate productivity is $\ln A_t = \ln\gamma + \ln A_{t-1} + \ln z_t$, and $\hat{y}_t$ is output relative to this trend in aggregate productivity.

on $\epsilon_{m,t}$

$$\hat{m}_t^{LL} = -\frac{\ln(1-\rho_m L)}{\rho_m L}\epsilon_{m,t},\ m \in \{g,z\} \tag{122}$$

and a harmonic lag operator specification, a modification of the log lag transfer function involving harmonic series - see Spiegel, Lipschutz, and Liu (2009, p. 141) - and to have a unit coefficient on $\epsilon_{m,t}$

$$\hat{m}_t^{LHL} = -\frac{\ln(1-\rho_m L)}{(1-\rho_m L)\rho_m L}\epsilon_{m,t},\ m \in \{g,z\} \tag{123}$$

The power series representation of the log lag specification - which translates to its MA(∞) representation - is

$$\hat{m}_t^{LL} = \sum_{j=0}^{\infty} \frac{\rho_m^j}{j}\mathscr{L}^j \epsilon_{m,t},\ m \in \{g,z\} \tag{124}$$

comparing this to the AR(1)

$$\hat{m}_t^{AR} = \sum_{j=0}^{\infty} \rho_m^j \mathscr{L}^j \epsilon_{m,t},\ m \in \{g,z\} \tag{125}$$

it is clear that the log lag specification has less autocorrelation for a common ρ_m than an AR(1) specification and decreases more quickly in the horizon as it divides by the horizon j. For the harmonic log, this is[19]

$$\hat{m}_t^{LHL} = \sum_{j=0}^{\infty}\left(\sum_{k=1}^{j+1}\frac{1}{k}\right)\rho_m^j \mathscr{L}^j \epsilon_{m,t},\ m \in \{g,z\} \tag{126}$$

or recognizing the harmonic number $H_j = \sum_{k=1}^{j}\frac{1}{k}$

$$\hat{m}_t^{LHL} = \sum_{j=0}^{\infty} H_{j+1}\rho_m^j \mathscr{L}^j \epsilon_{m,t},\ m \in \{g,z\} \tag{127}$$

and now it is apparent that the log harmonic lag specification has more autocorrelation for a common ρ_m than an AR(1) specification and decreases less quickly in the horizon as it multiplies with $H_{j+1} \leq 1$.

The model is estimated on the same US data from the Great Moderation on GDP growth, inflation, and the Federal Funds rate as in Herbst and Schorfheide (2015), which are related to the model's equations via

$$\begin{aligned}
YGR_t &= \gamma^{(Q)} + 100(\hat{y}_t - \hat{y}_{t-1} + \hat{z}_t) \\
INFL_t &= \pi^{(A)} + 400\hat{\pi}_t \\
INT_t &= \pi^{(A)} + r^{(\mathscr{A})} + 4\gamma^{(Q)} + 400\hat{R}_t.
\end{aligned}$$

[19]See the appendix.

Name	Domain	Density	Para (1)	Para (2)
τ	$\mathbb{R}^+$	Gamma	2.00	0.50
κ	$[0,1]$	Uniform	0.00	1.00
ψ_1	$\mathbb{R}^+$	Gamma	1.50	0.25
ψ_2	$\mathbb{R}^+$	Gamma	0.50	0.25
$r^{(A)}$	$\mathbb{R}^+$	Gamma	0.50	0.50
$\pi^{(A)}$	$\mathbb{R}^+$	Gamma	7.00	2.00
$\gamma^{(Q)}$	$\mathbb{R}$	Normal	0.40	0.20
ρ_R	$[0,1)$	Uniform	0.00	1.00
ρ_G	$(-1,1)^*$	Uniform	0.00	1.00
ρ_Z	$(-1,1)^*$	Uniform	0.00	1.00
σ_R	$\mathbb{R}^+$	InvGamma	0.40	4.00
σ_G	$\mathbb{R}^+$	InvGamma	1.00	4.00
σ_Z	$\mathbb{R}^+$	InvGamma	0.50	4.00

TABLE 1. Parameter priors.

For the MA(1) model, ρ_G and ρ_Z are improper priors over $\mathbb{R}$.

The parameters $\gamma^{(Q)}, \pi^{(A)}$, and $r^{(A)}$ are related to the steady states of the model economy as follows

$$\gamma = 1 + \frac{\gamma^{(Q)}}{100}, \quad \beta = \frac{1}{1 + r^{(A)}/400}, \quad \pi = 1 + \frac{\pi^{(A)}}{400}.$$

In the first-order approximation specified here, the parameters ν and ϕ are not separately identifiable so κ defined above will be taken as structural. The structural parameters are hence

$$\theta = \left[\tau, \kappa, \psi_1, \psi_2, \rho_R, \rho_g, \rho_z, r^{(\mathscr{A})}, \pi^{(A)}, \gamma^{(Q)}, \sigma_R, \sigma_g, \sigma_z\right]$$

The priors are contained in table 1. They are common across specifications and identical to those in Herbst and Schorfheide (2015). For the $MA(1)$ specification, note that invertibility or fundamentality is not necessary for stationarity and produces the same autocovariance function. I follow Meyer-Gohde and Neuhoff's (2018) Bayesian DSGE implementation of Lippi and Reichlin's (1994) root flipping for MA polynomials but - as the process is assumed first order - the standard Random Walk Metropolis Hastings sampling from the posterior will be sufficient to sample from the fundamental and non-fundamental representations.

Parameter	AR(1)	MA(1)	Log Lag	Log Harmonic Lag
τ	2.6236	3.1866	2.1936	3.0224
	(1.8117 3.5500)	(2.2481 4.2721)	(1.4622 3.0427)	(2.1233 4.0537)
κ	0.7730	0.6296	0.5655	0.8470
	(0.5184 0.9771)	(0.4576 0.8557)	(0.3149 0.8995)	(0.6348 0.9877)
ψ_1	1.9309	1.4719	1.7373	1.8620
	(1.5134 2.3656)	(1.1309 1.8549)	(1.3687 2.1324)	(1.4624 2.2931)
ψ_2	0.7329	0.3044	0.7396	0.7526
	(0.2655 1.3732)	(0.1352 0.5171)	(0.2768 1.3503)	(0.2798 1.3767)
$r^{(A)}$	1.4978	1.1917	1.6279	1.5395
	(1.0070 2.0678)	(0.8557 1.5567)	(1.0910 2.2415)	(1.0267 2.1455)
$\pi^{(A)}$	3.5926	3.1167	3.8864	3.3986
	(3.0334 4.2301)	(2.8619 3.3721)	(3.1369 4.7810)	(2.9441 3.8847)
$\gamma^{(Q)}$	0.5136	0.5312	0.5213	0.4350
	(0.4269 0.5944)	(0.5038 0.5586)	(0.4901 0.5523)	(0.2570 0.5679)
ρ_r	0.7985	0.6623	0.8132	0.7849
	(0.7382 0.8517)	(0.5781 0.7356)	(0.7455 0.8723)	(0.7246 0.8372)
ρ_g	0.9819	- 0.8246	0.9799	0.9566
	(0.9514 0.9988)	-(0.9659 -0.6800)	(0.9439 0.9988)	(0.8859 0.9972)
ρ_z	0.8543	- 0.1786	0.9840	0.7392
	(0.7949 0.9095)	-(0.9505 0.8432)	(0.9531 0.9992)	(0.6810 0.7939)
σ_r	0.2100	0.2457	0.2055	0.2110
	(0.1733 0.2539)	(0.1998 0.3007)	(0.1686 0.2494)	(0.1746 0.2540)
σ_g	0.6180	1.0549	0.9621	0.5665
	(0.5389 0.7089)	(0.8983 1.2352)	(0.8345 1.1090)	(0.4910 0.6515)
σ_z	0.3046	3.9570	0.7213	0.2736
	(0.2224 0.4153)	(2.6003 5.8635)	(0.5070 1.0179)	(0.2058 0.3638)
Marg. Data Density	-326.3955	-437.3892	-364.8139	-324.0800

TABLE 2. Posterior mean parameter estimates, 5% and 95% credible set bounds in parantheses

The posteriors can be found in table 2.[20] For the structural or "deep" parameters, there is a broad consensus across specifications, consistent with their interpretation as structural and the values reported in Herbst and Schorfheide (2015). The parameters for

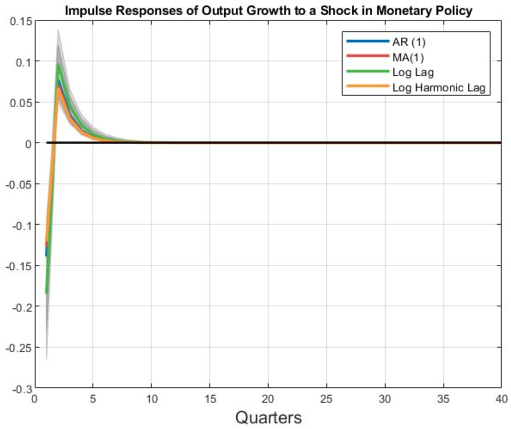
[20]The parameter posterior distributions can be found in hte appendix, likewise recursive averages that point to convergence of the Markov chains used to generate draws from the posteriors.

the exogenous processes are not comparable directly, of course, as the specifications are vastly different. At the bottom of the table, the marginal data density shows that the data indeed has a preference regarding the different specifications and while the standard AR(1) specification is certainly preferred over the MA(1) specification, the log harmonic lag improves on the AR(1) specification by three log points. Hence among the four equally parsimonious exogenous processes examined here, I have convincing evidence that the standard AR(1) process is not the most favored by the data. Understanding why will be addressed in the following impulse and shock decomposition analyses.

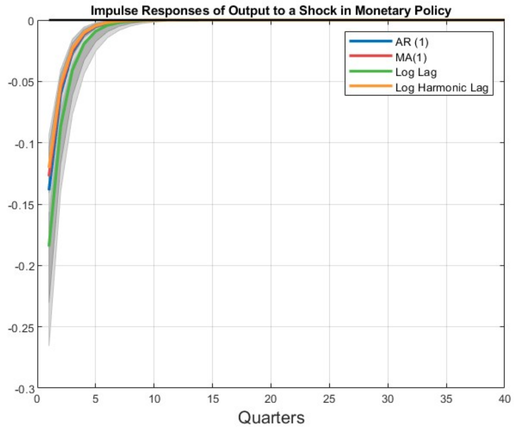
Turning to the posterior impulse response analysis and beginning with the response to a monetary policy shock, figure 1 shows almost perfect agreement among the four different specifications. The contractionary monetary policy shock leads output to fall below its natural level and inflation below its steady state level. While on the one hand we might expect this as the shock is assumed iid and not subject to the different specifications imposed on the remaining shocks, this provides evidence that the different specifications on the remaining shocks do not substantially affect the identification of the monetary policy shock.

Figure 2 depicts the impulse responses to a technology shock. Here we see the effects of the four different specifications for the exogenous processes very clearly. Beginning in the lower left with figure 2e, the MA(1) specification, due to the sampling from the root flipping at the posterior draws of the MA(1) parameter, demonstrates a very wide response containing even negative on impact, news shock[21] like specifications - see Meyer-Gohde and Neuhoff (2018) for higher order ARMA specifications in an RBC model. Figure 2f zooms in on the remaining three specifications. The standard AR(1) specification is highly persistent, a pervasive feature dating back to Kydland and Prescott (1982). Turning to the new specifications I introduce to the literature, the log lag specification (122) has a parameter estimate of ρ_z of 0.9840 at the posterior mode and nonetheless fails to generate as much persistence as the AR(1) specification. This was to be expected from above and to generate sufficient volatility in the productivity process, the standard deviation of

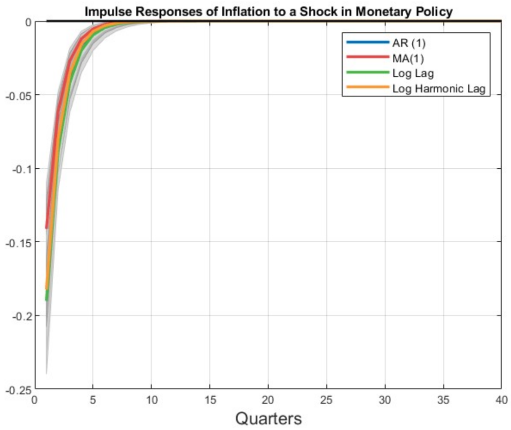
[21] Anticipated productivity movements as drivers of the business cycle have been examined frequently, with Barsky, Basu, and Lee (2015) and Portier (2015) providing an interim critical assessment. This paradigm has been extended recently to models of credit frictions, Görtz, Tsoukalas, and Zanetti (2022), and labor market frictions, Chahrour, Chugh, and Potter (2023), among others. The perspective here is perhaps closest to Schmitt-Grohé and Uribe (2012) who examine pervasive news shocks and find they contribute significantly to the business cycle.



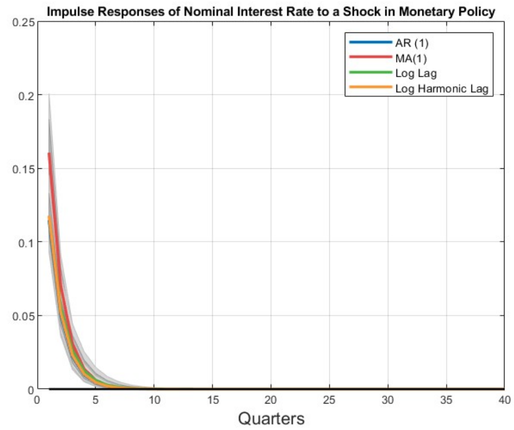
(A) Output Growth



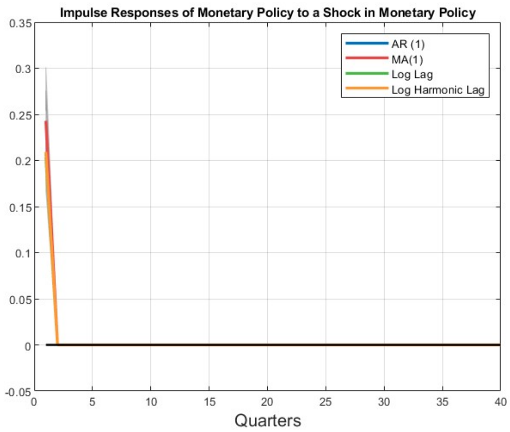
(B) Output



(C) Inflation



(D) Nominal Interest Rate

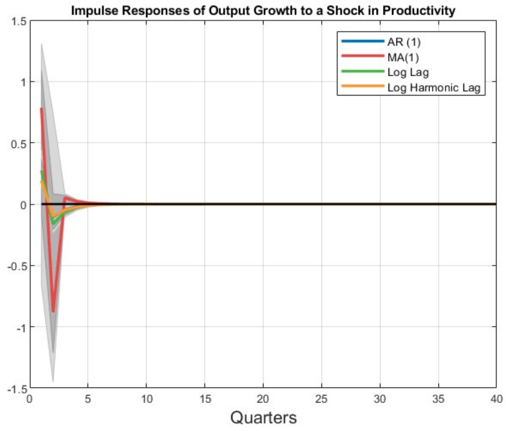


(E) Monetary Policy Shock

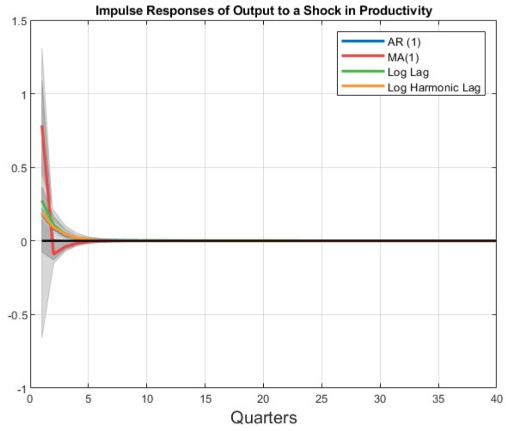
FIGURE 1. Monetary Policy Shock

the shock is estimated to be higher than in the AR(1) specification, as is reflected in the magnitude of the shock on impact. The log harmonic lag specification (122) is estimated to be highly persistent like the AR(1) specification, but generates a more complex autocorrelation pattern than the exponential decay that the AR(1) is limited too. This hump-shaped pattern, with the largest effect coming not on impact combines the flavor of a non invertible MA(1) representation with the sustained autocorrelation of an AR(1) process. In terms of the responses of the endogenous variables, the AR(1), log lag, and log harmonic lag all provide remarkably similar responses, with positive and persistent responses of output and inflation, consistent with monetary policy's contractionary response by raising interest rates. Note the hump-shaped dynamics in inflation for all three of the non MA specifications, generated by the interplay of the substantial interest rate smoothing at the persistent shock. While the log lag specification does not predict as much persistence at lower horizons as do the AR(1) and log harmonic lag ones, it does imply higher persistence at higher lags with both the nominal interest rate and inflation being noticeably non zero even ten years after the shock

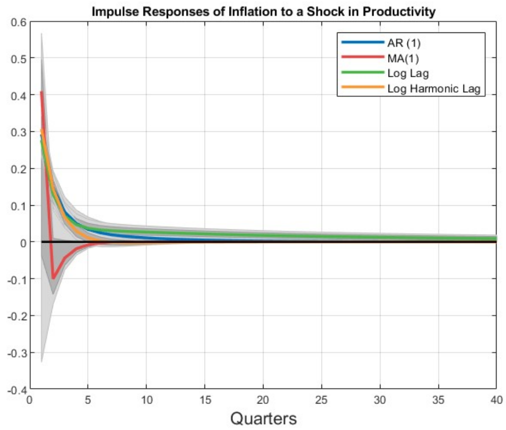
Figure 3 depicts the impulse responses to a government expenditure shock. Beginning with the response of government expenditures in figure 3e, the differences between the four specifications are dramatically apparent. While the MA(1) specification provides limited persistence, it does contain non-fundamental responses that, while not as dramatic as for technology, lead to wide posterior cedible sets on the responses. The log lag specification predicts a faster than exponential decrease in the response initially that however become very persistent at higher horizons. The AR(1) specification is very highly autocorrelated but like the log lag specification monotonic in its response. Very different is the hump shaped pattern produced by the log harmonic lag specification, combining the news shock later peak response of non fundamental MA processes with the sustained persistence of an AR process. This pattern is passed to output, while inflation and the nominal interest rate respond to the output gap $y_t - g_t$ which remains unchanged in this simple model. While not as visually striking as the response of output, the response of output growth is decisive and likely responsible for the higher log posterior data density of the log harmonic lag specification, as only it predicts a monotone response of output growth to government expenditures and it is the only response of output growth to any shock that displays this pattern. This gives output growth the autocorrelation needed to match the data for the log harmonic lag specification, with variation in this shock



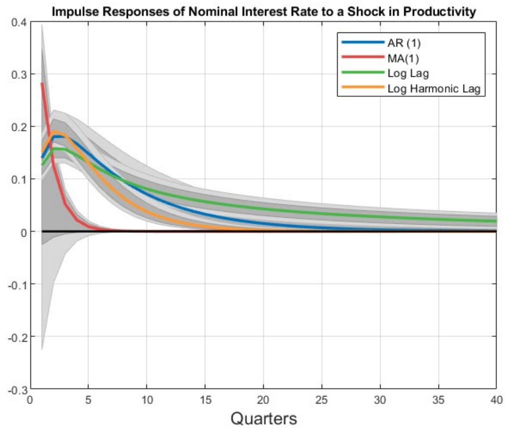
(A) Output Growth



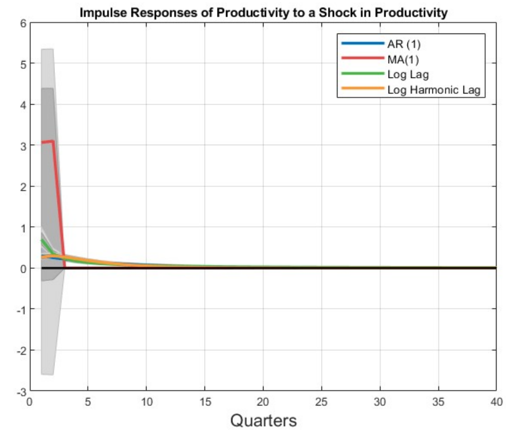
(B) Output



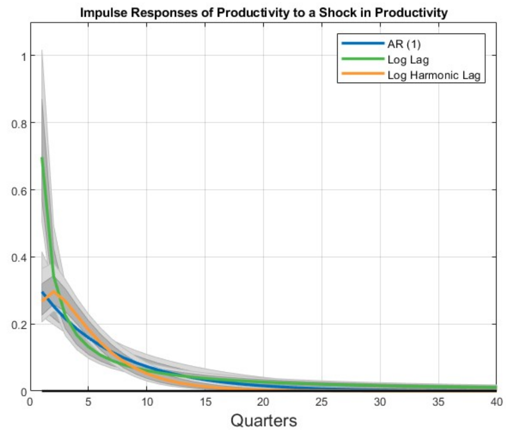
(C) Inflation



(D) Nominal Interest Rate



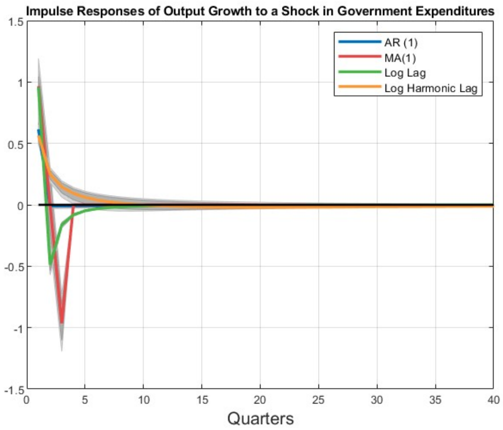
(E) Technology



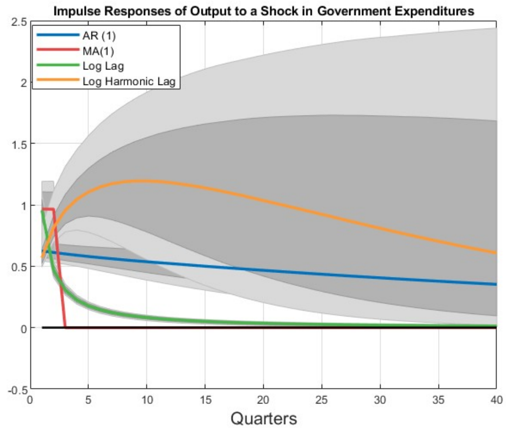
(F) Technology - w/o MA(1)

FIGURE 2. Technology Shock

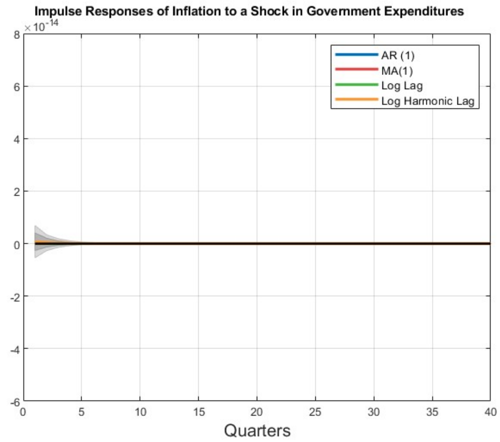
responsible for the majority of the variation in output growth according to the variance decompositions that follow, and is responsible for why all three other specifications fail to do so.



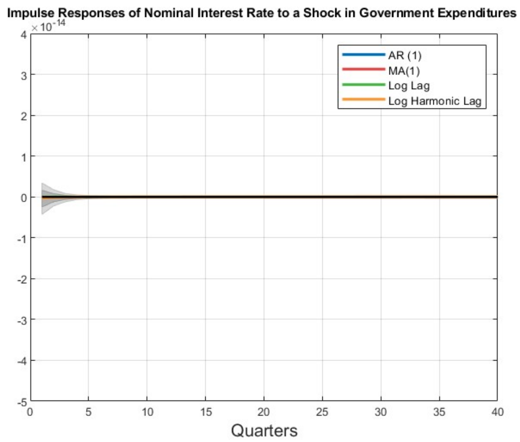
(A) Output Growth



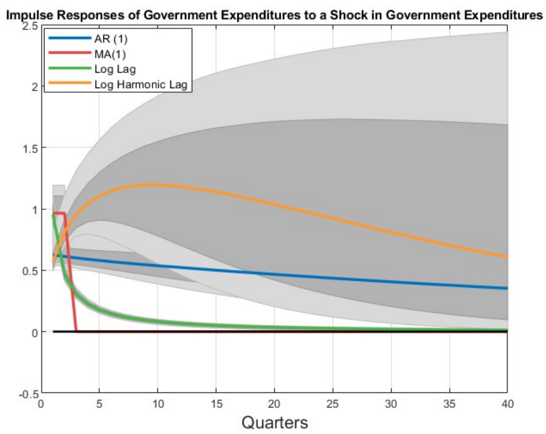
(B) Output



(C) Inflation



(D) Nominal Interest Rate



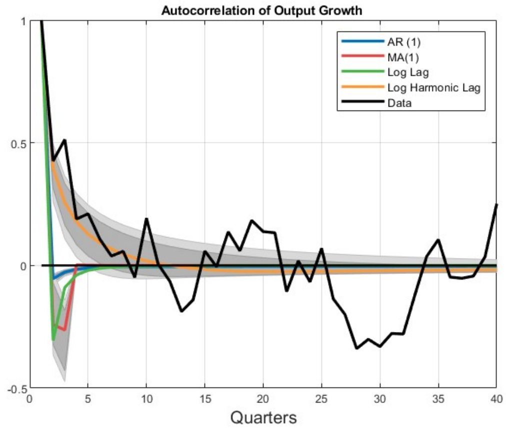
(E) Government Expenditure

FIGURE 3. Government Expenditure Shock

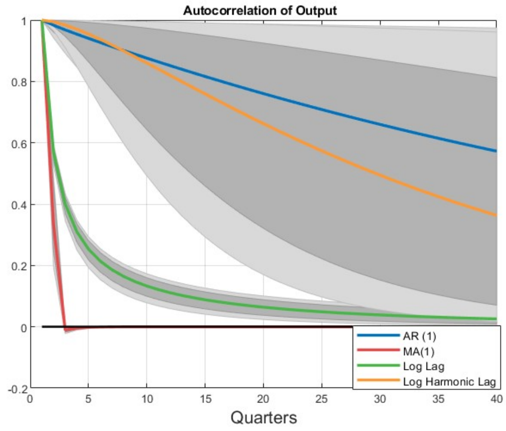
In figure 4, the posterior autocorrelations of variables along with, when available, the empirical counterparts are plotted. The absence of sustained autocorrelation under the MA(1) specification for technology and government expenditures leads to this specification significantly underestimating the persistence of variables, relative to the data and the remaining specifications. With respect to inflation and the nominal interest rate, the three remaining methods perform similarly, with the log lag version suggesting very significant autocorrelation at high horizons, something not possible with the exponential decay inherited under the AR(1) specifications. As a consequence of their autocorrelation patterns for government expenditures and the dominance on this shock on output - see both the impulses above and the variance decompositions below - the AR(1) and log harmonic lag both predict highly persistent outputs.

Finally, table 3 contains the variance decomposition of the four endogenous variables under the four specifications. Starting at the bottom with output growth, the AR(1) and both log lag specifications roughly agree with government expenditures driving approximately 85%, technology 10%, and monetary policy 5% of its variation. The MA(1) specification attributes an approximately 50-50 split between government expenditures and technology. All specifications are in general agrement that variations in the nominal interest rate and inflation are driven primarily by technology shocks and then by monetary policy shocks. The MA(1) specification, however, attributes a significantly higher proportion of the variation in interest rates and a lower proportion in inflation to monetary policy shocks than the other methods, which would seem consistent with the reduction in the systematic inflation stabilization in the policy rule, with other sources apart from monetary policy shocks - deviations from the systemic component - responsible for move of the movements in inflation. According to both the AR(1) and log harmonic lag specifications, variations in output are almost entirely driven by government expenditure shocks this follows not because technology and monetary policy shocks have no effect on output, but rather because government expenditure shocks have such a disproportionately large effect, see the impulse responses above.

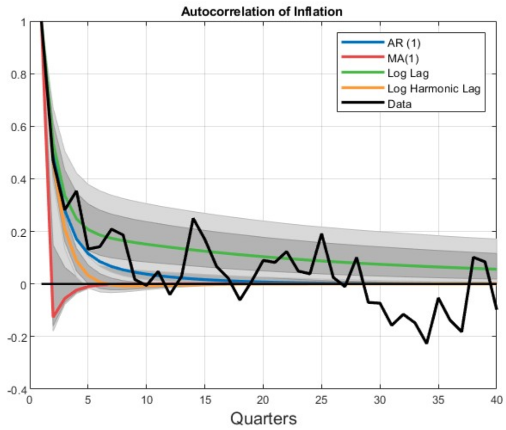
The frequency domain solution method of the foregoing sections enables the analysis of exogenous driving forces beyond the standard ARMA types that lead to transfer functions as ratios of polynomials or rational functions of the lag operator. Specifically I estimated and analyzed the effects of log lag and log harmonic lag operator transfer functions in a canonical New Keynesian model and find that the data prefers the log harmonic



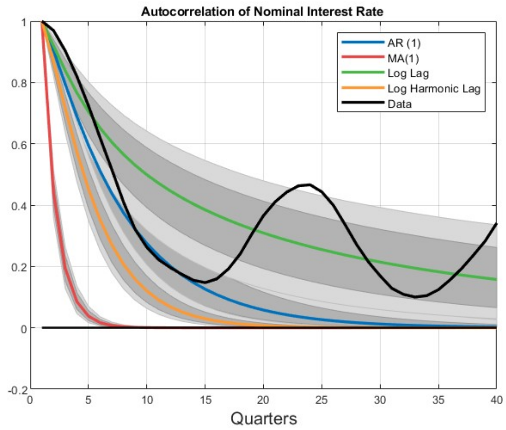
(A) Output Growth



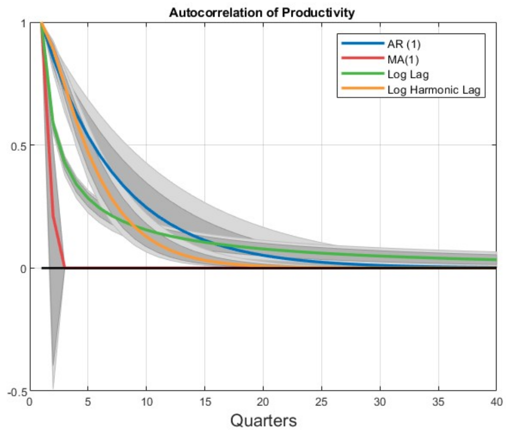
(B) Output



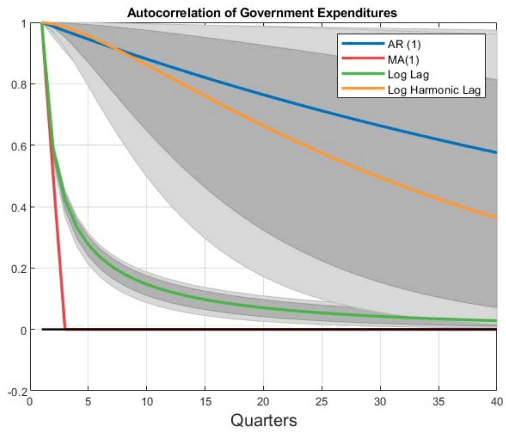
(C) Inflation



(D) Nominal Interest Rate



(E) Technology



(F) Government Expenditure

FIGURE 4. Autocorrelation

	Technology	Gov't Expenditures	Monetary Policy
		Output	
AR(1)	0.0033	0.9949	0.0017
	(0.0002 0.0142)	(0.9792 0.9998)	(0.0001 0.0069)
MA(1)	0.2572	0.7350	0.0079
	(0.0036 0.4944)	(0.4979 0.9850)	(0.0043 0.0155)
Log Lag	0.0573	0.9126	0.0284
	(0.0210 0.1449)	(0.8033 0.9637)	(0.0132 0.0574)
Log Harmonic Lag	0.0011	0.9985	0.0004
	(0.0000 0.0086)	(0.9884 1.0000)	(0.0000 0.0031)
		Inflation	
AR(1)	0.7274	0.0000	0.2726
	(0.6007 0.8373)	(0.0000 0.0000)	(0.1627 0.3993)
MA(1)	0.8798	0.0000	0.1202
	(0.0638 0.9388)	(0.0000 0.0000)	(0.0612 0.9362)
Log Lag	0.7166	0.0000	0.2834
	(0.5775 0.8362)	(0.0000 0.0000)	(0.1638 0.4225)
Log Harmonic Lag	0.7440	0.0000	0.2560
	(0.6345 0.8364)	(0.0000 0.0000)	(0.1636 0.3655)
		Nominal Interest Rate	
AR(1)	0.9249	0.0000	0.0751
	(0.8628 0.9632)	(0.0000 0.0000)	(0.0368 0.1372)
MA(1)	0.7492	0.0000	0.2508
	(0.0286 0.8758)	(0.0000 0.0000)	(0.1242 0.9714)
Log Lag	0.9227	0.0000	0.0773
	(0.8630 0.9565)	(0.0000 0.0000)	(0.0435 0.1370)
Log Harmonic Lag	0.9043	0.0000	0.0957
	(0.8390 0.9466)	(0.0000 0.0000)	(0.0534 0.1610)
		Output Growth	
AR(1)	0.1064	0.8329	0.0583
	(0.0603 0.1872)	(0.7332 0.8967)	(0.0345 0.0970)
MA(1)	0.4293	0.5643	0.0066
	(0.0076 0.6767)	(0.3179 0.9795)	(0.0033 0.0158)
Log Lag	0.0790	0.8848	0.0341
	(0.0317 0.1780)	(0.7689 0.9471)	(0.0171 0.0646)
Log Harmonic Lag	0.0972	0.8613	0.0397
	(0.0591 0.1600)	(0.7846 0.9113)	(0.0231 0.0668)

TABLE 3. Posterior variance decompositions, 5% and 95% credible set bounds in parentheses

lag specification, particularly because its hump shaped dynamics in the government expenditure process yields a persistently positive response of output growth to government expenditure shocks, enabling the model to recreate the sustained positive autocorrelation in output growth found in the data.

8. Conclusion

I have shown that the familiar QZ algorithm of generalized Schur decomposition can be applied to the solution of multivariate DSGE models in the frequency domain. The triangularization of the endogenous lead-lag structure allows Cauchy's residue theorem to be applied recursively to continue the transfer functions over the singularities on the unit disk induced by unstable eigenvalues. This recovers the missing initial conditions or the initial response of forward-looking or jump variables consistent with the model being on the stable arm of the system. I show how the solution can be assembled into a multivariate lag operator factorization that connects to univariate representations from Sargent's (1987, Ch. XIV) and makes the application of inverse Fourier transforms to recover impulse responses, autocovariance functions, and consequently the likelihood function straightforward.

In an application to the canonical New Keynesian model, I compare two transfer functions that non rational functions of the lag operator (specifically involving the logarithm of the lag operator) to polynomial specifications, i.e., AR and MA processes, for the exogenous driving forces. I find that the data prefers the log harmonic function of the lag operator over the standard AR(1) as the former is able to induce hump-shaped dynamics and more closely match the autocorrelation patterns in the data, especially output growth. While not preferred by the data, the other non rational transfer function implies autocorrelation patterns for endogenous variables that do not decay exponentially, instead persisting consistent with, say, long memory following Granger and Joyeux (1980) and Hassler (2019).

References

Adjemian, S., H. Bastani, M. Juillard, F. Mihoubi, G. Perendia, M. Ratto, and S. Villemot (2011): "Dynare: Reference Manual, Version 4," Dynare Working Papers 1, CEPREMAP.

Ahlfors, L. V. (1979): *Complex Analysis: An Introduction to The Theory of Analytic Functions of One Complex Variable*. McGraw-Hill, 3rd edn.

Al-Sadoon, M. M. (2020): "The Spectral Approach to Linear Rational Expectations Models," Papers 2007.13804, arXiv.org.

Barsky, R. B., S. Basu, and K. Lee (2015): "Whither News Shocks?," *NBER Macroeconomics Annual*, 29, 225–264.

BLANCHARD, O. J. (1979): “Backward and Forward Solutions for Economies with Rational Expectations,” *The American Economic Review*, 69(2), 114–118.

BLANCHARD, O. J., AND C. M. KAHN (1980): “The Solution of Linear Difference Models under Rational Expectations,” *Econometrica*, 48(5), 1305–1311.

CHAHROUR, R., S. K. CHUGH, AND T. POTTER (2023): “Anticipated productivity and the labor market,” *Quantitative Economics*, 14(3), 897–934.

COGLEY, T., AND J. M. NASON (1995): “Output Dynamics in Real-Business-Cycle Models,” *American Economic Review*, 85(3), 492–511.

DENNIS, JR., J. E., J. F. TRAUB, AND R. P. WEBER (1976): “The Algebraic Theory of Matrix Polynomials,” *SIAM Journal on Numerical Analysis*, 13(6), 831–845.

DÜCK, A., AND F. VERONA (2023): “Robust frequency-based monetary policy rules,” IMFS Working Paper Series 180, Goethe University Frankfurt, Institute for Monetary and Financial Stability (IMFS).

FERNÁNDEZ-VILLAVERDE, J., J. RUBIO-RAMÍREZ, AND F. SCHORFHEIDE (2016): “Solution and Estimation Methods for DSGE Models,” in *Handbook of Macroeconomics, Volume 2A*, ed. by J. B. Taylor, and H. Uhlig, Handbooks in Economics, chap. 9, pp. 527–724. North-Holland/Elsevier.

FUTIA, C. A. (1981): “Rational Expectations in Stationary Linear Models,” *Econometrica*, 49(1), 171–92.

GANTMACHER, F. R. (1959): *The Theory of Matrices*, vol. I&II. Chelsea Publishing Company, New York, NY.

GOLUB, G. H., AND C. F. VAN LOAN (1996): *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, and London, UK, 3 edn.

GÖRTZ, C., J. D. TSOUKALAS, AND F. ZANETTI (2022): “News shocks under financial frictions,” *American Economic Journal: Macroeconomics*, 14(4), 210–243.

GRANGER, C. W. J., AND R. JOYEUX (1980): “An introduction to long-memory time series models and fractional differencing,” *Journal of Time Series Analysis*, 1(1), 15–29.

HAMILTON, J. D. (1994): *Time Series Analysis*. Princeton University Press, Princeton.

HAN, Z., F. TAN, AND J. WU (2022): “Analytic policy function iteration,” *Journal of Economic Theory*, 200, 105395.

HANSEN, L. P., AND T. J. SARGENT (1980): “Formulating and estimating dynamic linear rational expectations models,” *Journal of Economic Dynamics and Control*, 2, 7–46.

——— (1981): “A note on Wiener-Kolmogorov prediction formulas for rational expectations models,” *Economics Letters*, 8(3), 255–260.

HASSLER, U. (2019): *Time Series Analysis with Long Memory in View*. Springer International Publishing.

HERBST, E., AND F. SCHORFHEIDE (2015): *Bayesian Estimation of DSGE Models*. Princeton University Press.

HIGHAM, N. J. (2008): *Functions of Matrices: Theory and Computation*. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA.

HIGHAM, N. J., AND H.-M. KIM (2000): “Numerical Analysis of a Quadratic Matrix Equation,” *IMA Journal of Numerical Analysis*, 20, 499–519.

HORN, R. A., AND C. R. JOHNSON (2013): *Matrix Analysis*. Cambridge University Press, Cambridge; New York, 2nd edn.

KASA, K. (2000): “Forecasting the Forecasts of Others in the Frequency Domain,” *Review of Economic Dynamics*, 3(4), 726–756.

KASA, K., C. H. WHITEMAN, AND T. B. WALKER (2011): “Heterogeneous Beliefs and Tests of Present Value Models,” Mimeo June.

KING, R. G., AND M. W. WATSON (1998): “The Solution of Singular Linear Difference Systems under Rational Expectations,” *International Economic Review*, 39(4), 1015–26.

KLEIN, P. (2000): “Using the Generalized Schur Form to Solve a Multivariate Linear Rational Expectations Model,” *Journal of Economic Dynamics and Control*, 24(10), 1405–1423.

KYDLAND, F. E., AND E. C. PRESCOTT (1982): “Time to Build and Aggregate Fluctuations,” *Econometrica*, 50(6), 1345–70.

LAN, H., AND A. MEYER-GOHDE (2012): “Existence and Uniqueness of Perturbation Solutions to DSGE Models,” Mimeo September.

LEEPER, E., AND T. WALKER (2011): “Information Flows and News Driven Business Cycles,” *Review of Economic Dynamics*, 14(1), 55–71.

LEEPER, E. M., S.-C. YANG, AND T. B. WALKER (2012): “Foresight and Information Flows,” Mimeo April.

LIPPI, M., AND L. REICHLIN (1994): "VAR analysis, nonfundamental representations, blaschke matrices," *Journal of Econometrics*, 63(1), 307–325.

LOISEL, O. (2022): "New Principles For Stabilization Policy," Working Papers 2022-16, Center for Research in Economics and Statistics.

MANKIW, N. G., AND R. REIS (2002): "Sticky Information versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve," *The Quarterly Journal of Economics*, 117(4), 1295–1328.

MARTINS, M. M., AND F. VERONA (2023): "Inflation dynamics in the frequency domain," *Economics Letters*, 231(C).

MEYER-GOHDE, A. (2010): "Linear Rational-Expectations Models with Lagged Expectations: A Synthetic Method," *Journal of Economic Dynamics and Control*, 34(5), 984–1002.

——— (2023): "Numerical Stability Analysis of Linear DSGE Models - Backward Errors, Forward Errors and Condition Numbers," IMFS Working Paper Series 193, Goethe University Frankfurt, Institute for Monetary and Financial Stability (IMFS).

MEYER-GOHDE, A., AND D. NEUHOFF (2015): "Solving and estimating linearized DSGE models with VARMA shock processes and filtered data," *Economics Letters*, 133, 89–91.

——— (2018): "Generalized exogenous processes in DSGE: A Bayesian approach," Discussion paper.

MEYER-GOHDE, A., AND M. TZAAWA-KRENZLER (2023): "Sticky Information and the Taylor Principle," IMFS Working Paper Series 189, Goethe University Frankfurt, Institute for Monetary and Financial Stability (IMFS).

——— (2024): "Inattentiveness and the Taylor Principle," Mimeo.

NELSON, C. R., AND C. R. PLOSSER (1982): "Trends and random walks in macroeconmic time series: Some evidence and implications," *Journal of Monetary Economics*, 10(2), 139–162.

ONATSKI, A. (2006): "Winding number criterion for existence and uniqueness of equilibrium in linear rational expectations models," *Journal of Economic Dynamics and Control*, 30(2), 323–345.

OPPENHEIM, A. V., R. W. SCHAFER, AND J. R. BUCK (1999): *Digital Signal Processing*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 2nd edn.

OPPENHEIM, A. V., A. S. WILLSKY, AND S. H. NAWAB (1996): *Signals & Systems (2nd Ed.)*. Prentice-Hall, Inc., USA.

PORTIER, F. (2015): "Comment," *NBER Macroeconomics Annual*, 29, 265–278.

PRIESTLY, M. B. (1981): *Spectral Analysis and Time Series*, Probability and mathematical statistics. Academic Press.

PROAKIS, J. G., AND D. G. MANOLAKIS (1996): *Digital signal processing: principles, algorithms, and applications*. Prentice-Hall, Inc., USA.

SALA, L. (2015): "DSGE Models in the Frequency Domain," *Journal of Applied Econometrics*, 30(2), 219–240.

SARGENT, T. J. (1987): *Macroeconomic Theory*. Academic Press, San Diego, CA, 2nd edn.

SCHMITT-GROHÉ, S., AND M. URIBE (2012): "What's News in Business Cycles," *Econometrica*, 80(6), 2733–2764.

SHUMWAY, R. H., AND D. S. STOFFER (2011): *Time Series Analysis and Its Applications With R Examples*. Springer, 3 edn.

SIMS, C. A. (2001): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1-2), 1–20.

SMETS, F., AND R. WOUTERS (2007): "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *The American Economic Review*, 97(3), 586–606.

SPIEGEL, M. R., S. LIPSCHUTZ, AND J. LIU (2009): *Mathematical Handbook of Formulas and Tables*. McGraw Hill, New York, 3rd edn.

TAN, F. (2021): "A Frequency-Domain Approach to Dynamic Macroeconomic Models," *Macroeconomic Dynamics*, 25(6), 1381–1411.

TAN, F., AND T. B. WALKER (2015): "Solving generalized multivariate linear rational expectations models," *Journal of Economic Dynamics and Control*, 60(C), 95–111.

TAYLOR, J. B. (1986): "Econometric Approaches to Stabilization Policy in Stochastic Models of Macroeconomic Fluctuations," in *Handbook of Econometrics*, ed. by Z. Griliches, and M. D. Intriligator, vol. 3 of *Handbook of Econometrics*, chap. 34, pp. 1997–2055. Elsevier.

UHLIG, H. (1999): "A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily," in *Computational Methods for the Study of Dynamic Economies*, ed. by R. Marimon,

and A. Scott, chap. 3, pp. 30–61. Oxford University Press.
VAN DOOREN, P. (2004): "The basics of developing numerical algorithms," *IEEE Control Systems Magazine*, 24(1), 18–27.
VILLEMOT, S. (2011): "Solving Rational Expectations Models at First Order: What Dynare Does," Dynare Working Papers 2, CEPREMAP.
WHITEMAN, C. H. (1983): *Linear Rational Expectations Models: A User's Guide*. University of Minnesota Press, Minneapolis, MN.
WHITEMAN, C. H., AND K. F. LEWIS (2010): "Prediction formulas," in *Macroeconometrics and Time Series Analysis*, ed. by S. N. Durlauf, and L. E. Blume, pp. 178–192. Palgrave Macmillan UK, London.

APPENDIX A. APPENDIX

FREQUENCY DOMAIN REPRESENTATION OF DISCRETE TIME SERIES

Here I present an (incomplete) introduction, following Priestly (1981), Ahlfors (1979), Oppenheim, Schafer, and Buck (1999, ch. 3), Oppenheim, Willsky, and Nawab (1996, ch. 10), Hamilton (1994, ch. 6), Sargent (1987, ch. XI), and Shumway and Stoffer (2011) to the z-transform and discrete time Fourier transform as it will pertain to our analysis of the determinacy of linear DSGE models. These transforms discern the frequency content and temporal dependencies of a given sequence and, hence, can be used in the analysis of discrete-time series.

My basic assumptions follow, e.g., Priestly (1981, ch. 4.11.) or Shumway and Stoffer (2011, Appendix C), for mean zero, linearly regular covariance stationary stochastic processes with absolutely continuous spectral distribution functions. Let y_t be such a process, then

$$y_t = \int_{-\pi}^{\pi} e^{it\omega} dZ(\omega) \tag{A-1}$$

where $dZ(\omega)$ is a mean zero, random orthogonal increment process with $E\left[|dZ(\omega)|^2\right] = h(\omega)d\omega$ and $E\left[dZ(\omega_1)dZ(\omega_2)^*\right] = 0,$ for $\omega_1 \neq \omega_2$. Assume that the autocovariance function is absolutely summable

$$\sum_{m=-\infty}^{\infty} \left|R_y(m)\right| < \infty \tag{A-2}$$

where the autocovariance function of a discrete-time series y_t is defined as

$$R_y(m) = \mathrm{Cov}(y_t, y_{t-m}) = E(y_t - \mu_y)(y_{t-m} - \mu_y) \tag{A-3}$$

then the spectral distribution function $Z(\omega)$ is absolutely continuous such that $dZ(\omega) = f_y(\omega)d\omega$ and $f_y(\omega)$ is the spectral density given by

$$f_y(\omega) = \sum_{m=-\infty}^{\infty} R_y(m)e^{-i\omega h}, \ -\pi \leq \omega \leq \pi \tag{A-4}$$

Whiteman (1983) assumes, and I follow, that solutions for y_t are sought in the space spanned by time-independent square-summable linear combinations of the process(es) fundamental for the driving process, that is H^2 or Hardy space.[22] Let ϵ_t be such a mean zero fundamental process with variance σ^2_ϵ. Its spectral density is thus

$$f_\epsilon(\omega) = \sum_{m=-\infty}^{\infty} R_\epsilon(m)e^{-i\omega h} = \frac{1}{2\pi}\sigma^2_\epsilon \tag{A-5}$$

[22]See, e.g., Han, Tan, and Wu (2022) for a more formal introduction.

Then an H^2 solution for an endogenous variable, y_t, is of the form $y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j}$ with $\sum_{j=0}^{\infty} y_j^2 < \infty$ and L the lag operator $Ly_t = y_{t-1}$.[23] Following, e.g., Sargent (1987, ch. XI) the Riesz-Fischer Theorem gives an equivalence (a one-to-one and onto transformation) between the space of squared summable sequences $\sum_{j=0}^{\infty} y_j^2 < \infty$ and the space of analytic functions in unit disk $y(z)$ corresponding to the z-transform of the sequence, $y(z) = \sum_{j=0}^{\infty} y_j z^j$.

Given a discrete series y_j with samples taken at equally spaced intervals, its z-transform $y(z)$ is defined in (2) as

$$y(z) = \sum_{j=0}^{\infty} y_j z^j \tag{A-6}$$

where z is a complex variable, and the sum extends from 0 to infinity, following the convention used in Hamilton (1994, ch. 6) and Sargent (1987, ch. XI).[24] By evaluating the z-transform on the unit circle in the complex plane ($z = e^{-i\omega}$, where ω is the angular frequency and i the complex number $\sqrt{-1}$), I obtain the discrete-time Fourier transform (DTFT). The DTFT $y(e^{-i\omega})$ is given by

$$y(e^{-i\omega}) = \sum_{j=0}^{\infty} y_j e^{-i\omega j} \tag{A-7}$$

The DTFT reveals the spectral characteristics of the sequence in terms of its frequency components.

The connection between the autocovariance function and the Fourier transformation of the z-transform evaluated on the unit circle ($z = e^{-i\omega}$) can be established by manipulating the equations

$$R_y(m) = \int_{-\pi}^{\pi} f_y(\omega) e^{im\omega} d\omega \tag{A-8}$$

Hence for our mean zero fundamental process ϵ_t

$$R_\epsilon(m) = \int_{-\pi}^{\pi} f_\epsilon(\omega) e^{im\omega} d\omega = \int_{-\pi}^{\pi} \frac{1}{2\pi} \sigma_\epsilon^2 e^{im\omega} d\omega = \frac{1}{2\pi} \sigma_\epsilon^2 \int_{-\pi}^{\pi} e^{im\omega} d\omega = \begin{cases} \sigma_\epsilon^2 & \text{for } m = 0 \\ 0 & \text{otherwise} \end{cases} \tag{A-9}$$

Now return to $y_t = y(L)\epsilon_t = \sum_{j=0}^{\infty} y_j \epsilon_{t-j}$ and recall $y_t = \int_{-\pi}^{\pi} e^{it\omega} dZ_y(\omega)$ and analogously $\epsilon_t = \int_{-\pi}^{\pi} e^{it\omega} dZ_\epsilon(\omega)$ so therefore it must hold that

$$\int_{-\pi}^{\pi} e^{it\omega} dZ_y(\omega) = \int_{-\pi}^{\pi} y(e^{it\omega}) e^{it\omega} dZ_\epsilon(\omega) \Rightarrow dZ_y(\omega) = y(e^{it\omega}) dZ_\epsilon(\omega) \tag{A-10}$$

[23]Note that I am abusing notation somewhat and choosing to use the same letter y to refer to a discrete time series, y_t, as well as that variable's transform function $y(z)$ or MA representation/response to a fundamental process j periods ago, y_j. This serves to save on the verbosity of notation, which might otherwise read $y_t = \sum_{j=0}^{\infty} \delta_j^y \epsilon_{t-j}$ following, e.g., Meyer-Gohde (2010).

[24]The discrete signal processing and systems theory literature works in negative exponents of z, see Oppenheim, Schafer, and Buck (1999, ch. 3) and Oppenheim, Willsky, and Nawab (1996, ch. 10). Al-Sadoon (2020) follows this convention and interprets the operator being applied as the forward operator. I maintain the more familiar approach in working with the lag operator which results in the use of positive exponents in z.

Multiplying both sides by their complex conjugates and taking expectations gives

$$E\left[dZ_y(\omega)dZ_y(\omega)^*\right] = E\left[y(e^{it\omega})y(e^{it\omega})^* dZ_\epsilon(\omega)dZ_\epsilon(\omega)^*\right] \tag{A-11}$$

$$f_y(\omega) = \left|y(e^{it\omega})\right|^2 f_\epsilon(\omega) = \left|y(e^{it\omega})\right|^2 \frac{1}{2\pi}\sigma_\epsilon^2 \tag{A-12}$$

I can insert this directly into (A-8) above to yield (4)

$$R_y(m) = \sigma_\epsilon^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|y(e^{-i\omega})\right|^2 e^{im\omega}d\omega \tag{A-13}$$

where $y(e^{-i\omega})$ and $y^*(e^{i\omega})$ denote the DTFT of y_j and its complex conjugate, respectively.

Proof of Theorem 4.14

Examining the final $j+1:n_y$ block of (50)

$$\left(S^+_{j+} - zT^+_{j+}\right)U_{j+}(z) = S^+_{j+}U_{j+}(0) - \tilde{W}_{j+}(z) \tag{A-14}$$

the pencil $S^+_{j+} - zT^+_{j+}$ has zeros at $\mu_{j+1},\ldots\mu_{n_y}$ but not at μ_j by virtue of assumption 4.13. Hence $\left(S^+_{j+} - \mu_j T^+_{j+}\right)$ is full rank and

$$U_{j+}(\mu_j) = \left(S^+_{j+} - \mu_j T^+_{j+}\right)^{-1}\left(S^+_{j+}U_{j+}(0) - \tilde{W}_{j+}(\mu_j)\right) \tag{A-15}$$

Turning now to the j'th row of (50)

$$\left(s_j - zt_j\right)u_j(z) + \left(S_{j+} - zT_{j+}\right)U_{j+}(z) = s_j u_j(0) + S_{j+}U_{j+}(0) - \tilde{w}_j(z) \tag{A-16}$$

which has a singularity in $\tilde{u}_j(z)$ inside the unit circle at $\mu_j \equiv s_j/t_j$ due to $\left(s_j - zt_j\right)$. Setting the residue to zero to remove the singularity gives

$$\lim_{z\to\mu_j}\left(s_j - zt_j\right)\tilde{u}_j(z) \stackrel{!}{=} 0 = -\left(S_{j+} - \mu_j T_{j+}\right)U_{j+}(\mu_j) + s_j u_j(0) + S_{j+}U_{j+}(0) - \tilde{w}_j(\mu_j) \tag{A-17}$$

Beginning at $j = n_y$ and proceeding to $j = 1$ gives the n_y elements of $U(0)$. Multiplying (48) with Z^*, the conjugate transpose of Z, gives, by virtue of unitary matrices,

$$\begin{bmatrix} Z^*_{11} & Z^*_{12} \\ Z^*_{21} & Z^*_{22}\end{bmatrix}\begin{bmatrix} zY(z) \\ Y(z)\end{bmatrix} = \begin{bmatrix} S(z) \\ U(z)\end{bmatrix} \tag{A-18}$$

solving the bottom block row for $Y(z)$ completes the proof.

Proof of Proposition 4.15

Assume the statement is true for $n-1$

$$(S_m - zT_m)U_m^{(n-1)}(z) - (n-1)T_m U_m^{(n-2)}(z) \tag{A-19}$$

$$+(S_{m+} - zT_{m+})U_{m+}^{(n-1)}(z) - (n-1)T_{m+}U_{m+}^{(n-2)}(z) = -\tilde{W}_m^{(n-1)}(z) \tag{A-20}$$

taking the derivative with respect to z yields

$$(S_m - zT_m)U_m^{(n)}(z) - T_m U_m^{(n-1)}(z) - (n-1)T_m U_m^{(n-1)}(z) \tag{A-21}$$

$$+(S_{m+} - zT_{m+})U_{m+}^{(n)}(z) - T_{m+}U_{m+}^{(n-1)}(z) - (n-1)T_{m+}U_{m+}^{(n-1)}(z) = -\tilde{W}_m^{(n)}(z) \tag{A-22}$$

Collecting terms in $T_m U_m^{(n-1)}(z)$ and $T_{m+} U_{m+}^{(n-1)}(z)$ delivers (81). Taking the derivative of (77) with respect to z delivers

$$(S_m - zT_m)U_m^{(1)}(z) - T_m U_m(z) + (S_{m+} - zT_{m+})U_{m+}(z) - T_{m+}U_{m+}^{(1)}(z) = -\tilde{W}_m^{(1)}(z) \tag{A-23}$$

which is (81) with $n = 1$.

Proof of Corollary 4.16

Examine (77) and (81) and recall from (78) that $S_m - zT_m$ is singular only for $z = \mu_m$, solving for $U_m^{(n)}(z)$ gives the results

Proof of Corollary 4.17

The n'th derivative of (77) with respect to z at $z = \mu_m$, (81) evaluated at $z = \mu_m$, is

$$\left(\hat{S}_m - \mu_m \hat{T}_m\right) U_m^{(n)}(\mu_m) - nT_m U_m^{(n-1)}(\mu_m) + \left(S_{m+} - \mu_m T_{m+}\right) U_{m+}^{(n)}(\mu_m) - nT_{m+}U_{m+}^{(n-1)}(\mu_m) = -\tilde{W}_m^{(n)}(\mu_m) \tag{A-24}$$

advancing the index from n to $n+1$ and solving for $\tilde{U}_m^{(n)}(\mu_m) \equiv (\hat{S}_m - \mu_m \hat{T}_m)U_m^{(n)}(\mu_m)$ gives

$$\tilde{U}_m^{(n)}(\mu_m) = \frac{1}{n+1}\Theta_m \left[\tilde{W}_m^{(n+1)}(\mu_m) + (S_{m^+} - \mu_m T_{m^+})U_{m^+}^{(n+1)}(\mu_m) - (n+1)T_{m^+}U_{m^+}^{(n)}(\mu_m)\right] \tag{A-25}$$

$$+ \frac{1}{n+1}\Theta_m \tilde{U}_n^{(n+1)}(\mu_m) \tag{A-26}$$

where $\Theta_m \equiv \left(\hat{S}_m - \mu_m \hat{T}_m\right) T_m^{-1}$ and hence, solving forward gives

$$\tilde{U}_m^{(0)}(\mu_m) = \sum_{j=1}^{k_m-1} \frac{1}{j!}\Theta_m^j \left[\tilde{W}_m^{(j)}(\mu_m) + (S_{m^+}\mu_m T_{m^+})U_{m^+}^{(j)}(\mu_m) - jT_{m^+}U_{m^+}^{(j-1)}(\mu_m)\right] \tag{A-27}$$

as $\Theta_m^{k_m}$ is nilpotent with $\Theta_m^{k_m} = 0$

Proof of Proposition 4.18

Begin with (77)

$$(S_m - zT_m)U_m(z) + (S_{m+} - zT_{m+})U_{m+}(z) = S_m U_m(0) + S_{m+}U_{m+}(0) - \tilde{W}_m(z) \tag{A-28}$$

and rewrite using $\hat{S}_m = S_m - \tilde{S}_m$ and $\hat{T}_m = T_m - \tilde{T}_m$

$$\left(\tilde{S}_m - z\tilde{T}_m\right) U_m(z) = S_m U_m(0) + S_{m+}U_{m+}(0) - \tilde{W}_m(z) - (S_{m+} - zT_{m+})U_{m+}(z) - \left(\hat{S}_m - z\hat{T}_m\right) U_m(z) \tag{A-29}$$

Demanding the residual be equal to zero gives

$$\lim_{z\to\mu_j} \left(\tilde{S}_m - z\tilde{T}_m\right) U_m(z) \overset{!}{=} 0 = S_m U_m(0) + S_{m+}U_{m+}(0) - \tilde{W}_m(z) - (S_{m+} - zT_{m+})U_{m+}(z) - \left(\hat{S}_m - z\hat{T}_m\right) U_m(z) \tag{A-30}$$

solving for $U_m(0)$, note that S_m is upper triangular with nonzero diagonal elements and hence nonsingular for all m except the "infinite" eigenvalue case considered seperately, gives

$$U_m(0) = S_m^{-1}\left[(S_{m^+} - \mu_m T_{m^+})U_{m^+}(\mu_m) - S_{m^+}U_{m^+}(0) + \tilde{W}_m(\mu)\right] + S_m^{-1}\left[(\hat{S}_m - \mu_m \hat{T}_m)U_m(\mu_m)\right] \tag{A-31}$$

recalling the definition of $\tilde{U}_m^{(n)}(\mu_m) \equiv (\hat{S}_m - \mu_m \hat{T}_m)U_m^{(n)}(\mu_m)$ from above gives the first equation in the proposition. Inserting (A-25) then the second.

Proof of Proposition 4.19

From (77), noting that block M is by definition the final block

$$(S_M - zT_M)U_M(z) = S_M U_M(0) - \tilde{W}_M(z) \tag{A-32}$$

which can be rearranged as

$$zT_M)U_M(z) = S_M(U_M(z) - U_M(0)) + \tilde{W}_M(z) - \tilde{W}_M(0) \tag{A-33}$$

as $\tilde{W}_M(0) = 0$. Developing further gives

$$U_M(z) = T_M^{-1}\hat{S}_M \frac{U_M(z) - U_M(0)}{z} + T_M^{-1}\frac{\tilde{W}_M(z) - \tilde{W}_M(0)}{z} \tag{A-34}$$

noting that if $\mu_M = 0$, the diagonal elements of S_M are zero and hence, $\hat{S}_M = S_M$. To determine $U_M(0)$, take the limit as z goes to $\mu_M = 0$

$$U_M(0) = T_M^{-1}\hat{S}_M \lim_{z\to 0}\frac{U_M(z) - U_M(0)}{z} + T_M^{-1}\lim_{z\to 0}\frac{\tilde{W}_M(z) - \tilde{W}_M(0)}{z} = T_M^{-1}\hat{S}_M U_M^{(1)}(0) + T_M^{-1}\tilde{W}_M^{(1)}(0) \tag{A-35}$$

From corollary 4.17, noting that block M is by definition the final block and $\mu_M = 0$, it follows that $\Theta_M \equiv \hat{S}_M T_M^{-1}$ and hence

$$\tilde{U}_M^{(1)}(0) = \sum_{j=1}^{k_M - 1}\frac{1}{(j+1)!}\Theta_M^{j+1}\left[\tilde{W}_M^{(j+1)}(0)\right] \tag{A-36}$$

with $\tilde{U}_M^{(n)}(\mu_m) \equiv \hat{S}_m U_m^{(n)}(0)$.

Combining yields

$$U_M(0) = T_M^{-1}\tilde{W}_M^{(1)}(0) + T_M^{-1}\sum_{j=1}^{k_M - 1}\frac{1}{(j+1)!}\Theta_M^{j+1}\left[\tilde{W}_M^{(j+1)}(0)\right] \tag{A-37}$$

or

$$U_M(0) = T_M^{-1}\sum_{j=0}^{k_M - 1}\frac{1}{(j+1)!}\Theta_M^{j+1}\left[\tilde{W}_M^{(j+1)}(0)\right] \tag{A-38}$$

Proof of Theorem 4.20

This follows directly from propositions 4.18 and 4.19.

Proof of Corollary 5.1

To prove the equivalence stated in the corollary, I first have to lay out some properties of unitary matrices, how they apply in the context of the QZ decomposition here, (46), and what restrictions are imposed by the decomposition via the partition of the blocks of Q, Z, S and T associated with the eigenvalues inside and outside the unit circle.

Proposition A.1. *For a unitary matrix* $Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ *and its conjugate transpose* $Z^* = \begin{bmatrix} Z^*_{11} & Z^*_{12} \\ Z^*_{21} & Z^*_{22} \end{bmatrix}$ *if* Z_{11} *is nonsingular, then so are* Z^*_{11}*,* Z_{22}*, and* Z^*_{22}*. Furthermore,* $Z^*_{11} = \left(Z_{11} + Z_{12}Z_{22}^{-1}Z_{21}\right)^{-1}$ *and* $Z^*_{22} = \left(Z_2 + Z_{21}Z_{11}^{-1}Z_{12}\right)^{-1}$*.*

Proof. This follows from the nonsingularity of unitary matrices, the principle pivot (Schur complement), and the equality of the conjugate transpose and inverse of a unitary matrix. □

Using 45 and 49

$$GZ = QS, \ HZ = QT \tag{A-39}$$

can be written as

$$\begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \tag{A-40}$$

$$\begin{bmatrix} 0 & I \\ -C & -B \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \tag{A-41}$$

which delivers the following eight equalities

$$\begin{bmatrix} Z_{11} & Z_{12} \\ AZ_{21} & AZ_{22} \end{bmatrix} = \begin{bmatrix} Q_{11}S_{11} & Q_{11}S_{12} + Q_{12}S_{22} \\ Q_{21}S_{11} & Q_{21}S_{12} + Q_{22}S_{22} \end{bmatrix} \tag{A-42}$$

$$\begin{bmatrix} Z_{21} & Z_{22} \\ -CZ_{11} - BZ_{21} & -CZ_{12} - BZ_{22} \end{bmatrix} = \begin{bmatrix} Q_{11}T_{11} & Q_{11}T_{12} + Q_{12}T_{22} \\ Q_{21}T_{11} & Q_{21}T_{12} + Q_{22}T_{22} \end{bmatrix} \tag{A-43}$$

Using these relations and the assumptions (4.5) and (4.6)

Proposition A.2. *If* Z_{11} *is nonsingular, then* Q_{11} *is nonsingular.*

Proof. The upper-left block equation for S in is $Q_{11}S_{11} = Z_{11}$. As the upper-triangular pencil $P(S_{11}, T_{11})$ has a full set of eigenvalues all of which are inside the unit circle, S_{11} is upper triangular with necessarily non-zero diagonal entries and therefore non-singular. □

From proposition A.1, all the results there apply to Q as well with the nonsingularity of Q_{11} from the foregoing. I know have the results to triangularize the solution to (30) in theorem 4.8

$$\left(A\frac{1}{\mathscr{B}} + AX + B\right)(I - X\mathscr{B})E_t[Y_t] = E_t[W_t] \tag{A-44}$$

Proposition A.3. *The following holds*

(1) $X = -{Z^*_{22}}^{-1}Z^*_{21} = Z_{21}{Z_{11}}^{-1} = Q_{11}{S_{11}}^{-1}T_{11}{Q_{11}}^{-1}$
(2) $A = {Q^*_{22}}^{-1}S_{22}Z^*_{22}$

(3) $AX + B = -{Q^*_{22}}^{-1} T_{22} Z^*_{22}$

Proof. Beginning with the first claim $X = -{Z^*_{22}}^{-1} Z^*_{21} = Z_{21} {Z_{11}}^{-1} = Q_{11} {S_{11}}^{-1} T_{11} {Q_{11}}^{-1}$. The solvent X for a given sorting of the generalized Schur deomposition associated with the companion linearization (45) is $Z_{21} {Z_{11}}^{-1}$. For the first equality $-{Z^*_{22}}^{-1} Z^*_{21} = Z_{21} {Z_{11}}^{-1}$ the results follow from Schur complements and proposition A.1

$$Z_{21} {Z_{11}}^{-1} = Z_{21} \left(Z^*_{11} - Z^*_{12} {Z^*_{22}}^{-1} Z^*_{21} \right) \tag{A-45}$$

$$= Z_{21} \left(Z^*_{11} + {Z_{11}}^{-1} Z_{12} Z^*_{21} \right) \tag{A-46}$$

$$= Z_{21} Z^*_{11} + Z_{21} {Z_{11}}^{-1} Z_{12} Z^*_{21} \tag{A-47}$$

$$= -Z_{22} Z^*_{21} + Z_{21} {Z_{11}}^{-1} Z_{12} Z^*_{21} \tag{A-48}$$

$$= -\left(Z_{22} - Z_{21} {Z_{11}}^{-1} Z_{12} \right) Z^*_{21} \tag{A-49}$$

$$= -{Z^*_{22}}^{-1} Z^*_{21} \tag{A-50}$$

and the second equality $Z_{21} {Z_{11}}^{-1} = Q_{11} {S_{11}}^{-1} T_{11} {Q_{11}}^{-1}$ follows from (A-42)

$$Z_{21} {Z_{11}}^{-1} = Q_{11} {S_{11}}^{-1} {Z_{11}}^{-1} \tag{A-51}$$

$$Z_{21} {Z_{11}}^{-1} = Q_{11} {S_{11}}^{-1} \left(Q_{11} {T_{11}}^{-1} \right)^{-1} \tag{A-52}$$

$$Z_{21} {Z_{11}}^{-1} = Q_{11} {S_{11}}^{-1} T_{11} {Q_{11}}^{-1} \tag{A-53}$$

Now for the second claim $A = {Q^*_{22}}^{-1} S_{22} Z^*_{22}$

$$A {Z^*_{22}}^{-1} = A \left(Z_{22} - Z_{21} {Z_{11}}^{-1} Z_{12} \right) \tag{A-54}$$

$$= Q_{21} S_{12} + Q_{22} S_{22} - A Z_{21} {Z_{11}}^{-1} Z_{12} \tag{A-55}$$

$$= Q_{21} S_{12} + Q_{22} S_{22} - Q_{21} S_{11} {Z_{11}}^{-1} Z_{12} \tag{A-56}$$

$$= Q_{21} \left(S_{12} - S_{11} {S_{11}}^{-1} {Q_{11}}^{-1} Z_{12} \right) + Q_{22} S_{22} \tag{A-57}$$

$$= Q_{21} {Q_{11}}^{-1} \left(Q_{11} S_{12} - Z_{12} \right) + Q_{22} S_{22} \tag{A-58}$$

$$= -Q_{21} {Q_{11}}^{-1} Q_{12} S_{22} + Q_{22} S_{22} \tag{A-59}$$

$$= \left(Q_{22} - Q_{21} {Q_{11}}^{-1} Q_{12} \right) S_{22} \tag{A-60}$$

$$= {Q^*_{22}}^{-1} S_{22} \tag{A-61}$$

$$\tag{A-62}$$

So $A {Z^*_{22}}^{-1} = {Q^*_{22}}^{-1} S_{22}$ and hence $A = {Q^*_{22}}^{-1} S_{22} Z^*_{22}$.

And, finally, $AX + B = -{Q^*_{22}}^{-1} T_{22} Z^*_{22}$

$$(AX + B) {Z^*_{22}}^{-1} = AX {Z^*_{22}}^{-1} + B {Z^*_{22}}^{-1} \tag{A-63}$$

$$= A Z_{21} {Z_{11}}^{-1} {Z^*_{22}}^{-1} + B {Z^*_{22}}^{-1} \tag{A-64}$$

$$= AZ_{21}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} + B\left(Z_{22} - Z_{21}Z_{11}{}^{-1}Z_{12}\right) \tag{A-65}$$

$$= AZ_{21}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} - CZ_{12} - Q_{21}T_{12} - Q_{22}T_{22} + CZ_{11}Z_{11}{}^{-1}Z_{12} + Q_{21}T_{11}Z_{11}{}^{-1}Z_{12} \tag{A-66}$$

$$= AZ_{21}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} - Q_{21}T_{12} - Q_{22}T_{22} + Q_{21}T_{11}Z_{11}{}^{-1}Z_{12} \tag{A-67}$$

$$= AZ_{21}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} + Q_{21}\left(T_{11}Z_{11}{}^{-1}Z_{12} - T_{12}\right) - Q_{22}T_{22} \tag{A-68}$$

$$= Q_{21}S_{11}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} + Q_{21}\left(T_{11}Z_{11}{}^{-1}Z_{12} - T_{12}\right) - Q_{22}T_{22} \tag{A-69}$$

$$= Q_{21}\left(S_{11}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} + T_{11}Z_{11}{}^{-1}Z_{12} - T_{12}\right) - Q_{22}T_{22} \tag{A-70}$$

$$= Q_{21}Q_{11}{}^{-1}\left(Q_{11}S_{11}Z_{11}{}^{-1}Z_{22}^{*}{}^{-1} + Q_{11}T_{11}Z_{11}{}^{-1}Z_{12} - Q_{11}T_{12}\right) - Q_{22}T_{22} \tag{A-71}$$

$$= Q_{21}Q_{11}{}^{-1}\left(Z_{22}^{*}{}^{-1} + Q_{11}T_{11}Z_{11}{}^{-1}Z_{12} - Q_{11}T_{12}\right) - Q_{22}T_{22} \tag{A-72}$$

$$= Q_{21}Q_{11}{}^{-1}\left(Z_{22} - Z_{21}Z_{11}{}^{-1}Z_{12} + Q_{11}T_{11}Z_{11}{}^{-1}Z_{12} - Q_{11}T_{12}\right) - Q_{22}T_{22} \tag{A-73}$$

$$= Q_{21}Q_{11}{}^{-1}\left(Z_{22} - Z_{21}Z_{11}{}^{-1}Z_{12} + Z_{21}Z_{11}{}^{-1}Z_{12} - Q_{11}T_{12}\right) - Q_{22}T_{22} \tag{A-74}$$

$$= Q_{21}Q_{11}{}^{-1}(Z_{22} - Q_{11}T_{12}) - Q_{22}T_{22} \tag{A-75}$$

$$= Q_{21}Q_{11}{}^{-1}Q_{12}T_{22} - Q_{22}T_{22} \tag{A-76}$$

$$= \left(Q_{21}Q_{11}{}^{-1}Q_{12} - Q_{22}\right)T_{22} \tag{A-77}$$

$$= Q_{22}^{*}{}^{-1}T_{22} \tag{A-78}$$

$$\tag{A-79}$$

So $(AX+B)Z_{22}^{*}{}^{-1} = Q_{22}^{*}{}^{-1}T_{22}$ and hence $AX+B = -Q_{22}^{*}{}^{-1}T_{22}Z_{22}^{*}$. □

Hence, (31)

$$\left(A\frac{1}{\mathscr{B}} + AX + B\right)(I - X\mathscr{B})Y_t = -E_t[W_t] \tag{A-80}$$

can be expressed as

$$\left(Q_{22}^{*}{}^{-1}S_{22}Z_{22}^{*}\frac{1}{\mathscr{B}} - Q_{22}^{*}{}^{-1}T_{22}Z_{22}^{*}\right)\left(I - Z_{21}Z_{11}{}^{-1}\mathscr{B}\right)Y_t = -E_t[W_t] \tag{A-81}$$

$$Q_{22}^{*}{}^{-1}\left(S_{22}Z_{22}^{*}\frac{1}{\mathscr{B}} - T_{22}Z_{22}^{*}\right)(Z_{11} - Z_{21}\mathscr{B})Z_{11}{}^{-1}Y_t = -E_t[W_t] \tag{A-82}$$

$$Q_{22}^{*}{}^{-1}\left(S_{22}\frac{1}{\mathscr{B}} - T_{22}\right)Z_{22}^{*}(Q_{11}S_{11} - Q_{11}T_{11}\mathscr{B})Z_{11}{}^{-1}Y_t = -E_t[W_t] \tag{A-83}$$

$$Q_{22}^{*}{}^{-1}\left(S_{22}\frac{1}{\mathscr{B}} - T_{22}\right)Z_{22}^{*}Q_{11}(S_{11} - T_{11}\mathscr{B})Z_{11}{}^{-1}Y_t = -E_t[W_t] \tag{A-84}$$

(A-85)

While the stable and unstable pencils are now visible in their triangular forms, $S_{11} - T_{11}\mathscr{B}$ and $S_{22}\frac{1}{\mathscr{B}} - T_{22}$ it is unclear whether the transformation is similar (i.e., eigenvalue preserving). To this end, I would need to be able to express $Z_{22}^{*}Q_{11}$ as RZ_{11} for some invertible R to have a similar transformation of the stable pencil.

To this end

$$\left(Z_{22}^{*}Q_{11}\right)^{-1} = Q_{11}^{-1}{Z_{22}^{*}}^{-1} = Q_{11}^{-1}\left(Z_{22} - Z_{21}{Z_{11}}^{-1}Z_{12}\right) \tag{A-86}$$

$$= Q_{11}^{-1}Z_{22} - Q_{11}^{-1}Z_{21}{Z_{11}}^{-1}Z_{12} \tag{A-87}$$

$$= Q_{11}^{-1}Z_{22} - Q_{11}^{-1}XZ_{12} \tag{A-88}$$

$$= Q_{11}^{-1}Z_{22} - Q_{11}^{-1}Q_{11}{S_{11}}^{-1}T_{11}{Q_{11}}^{-1}Z_{12} \tag{A-89}$$

$$= Q_{11}^{-1}Z_{22} - {S_{11}}^{-1}T_{11}{Q_{11}}^{-1}Z_{12} \tag{A-90}$$

$$= {S_{11}}^{-1}\left(S_{11}Q_{11}^{-1}Z_{22} - T_{11}{Q_{11}}^{-1}Z_{12}\right) \tag{A-91}$$

$$= {S_{11}}^{-1}Q_{11}^{-1}\left(Q_{11}S_{11}Q_{11}^{-1}Z_{22} - Q_{11}T_{11}{Q_{11}}^{-1}Z_{12}\right) \tag{A-92}$$

$$= (Q_{11}S_{11})^{-1}\left(Q_{11}S_{11}Q_{11}^{-1}Z_{22} - Q_{11}T_{11}{Q_{11}}^{-1}Z_{12}\right) \tag{A-93}$$

$$= Z_{11}^{-1}\left(\underbrace{Z_{11}Q_{11}^{-1}Z_{22} - Z_{21}{Q_{11}}^{-1}Z_{12}}_{R}\right) \tag{A-94}$$

And so $Z_{22}^{*}Q_{11} = RZ_{11}$.

Hence proving the claims behind equations (105) (105) in the corollary.

All that remains to prove is the one-to-one equivalence via a $\mathscr{Z}$ transform to theorem 4.20. This follows by inspection if it can be shown that

(1) $R^{-1}Z_{11}S_{11}{Z_{11}}^{-1} = Z_{22}^{*}$
(2) $-R^{-1}Z_{11}T_{11}{Z_{11}}^{-1} = Z_{21}^{*}$

Beginning with the first equality $R^{-1}Z_{11}S_{11}{Z_{11}}^{-1} = Z_{22}^{*}$

$$Z_{11}{S_{11}}^{-1}{Z_{11}}^{-1}R = Z_{11}{S_{11}}^{-1}{Z_{11}}^{-1}\left(Z_{11}Q_{11}^{-1}Z_{22} - Z_{21}{Q_{11}}^{-1}Z_{12}\right) \tag{A-95}$$

$$= Z_{11}{S_{11}}^{-1}\left(Q_{11}^{-1}Z_{22} - {Z_{11}}^{-1}Z_{21}{Q_{11}}^{-1}Z_{12}\right) \tag{A-96}$$

$$= Q_{11}S_{11}{S_{11}}^{-1}\left(Q_{11}^{-1}Z_{22} - {Z_{11}}^{-1}Z_{21}{Q_{11}}^{-1}Z_{12}\right) \tag{A-97}$$

$$= Q_{11}\left(Q_{11}^{-1}Z_{22} - {Z_{11}}^{-1}Z_{21}{Q_{11}}^{-1}Z_{12}\right) \tag{A-98}$$

$$= Z_{22} - Q_{11}{Z_{11}}^{-1}Z_{21}{Q_{11}}^{-1}Z_{12} \tag{A-99}$$

$$= Z_{22} - Q_{11}{Z_{11}}^{-1}Q_{11}T_{11}{Q_{11}}^{-1}Z_{12} \tag{A-100}$$

$$= Z_{22} - Q_{11}{S_{11}}^{-1}T_{11}{Q_{11}}^{-1}Z_{12} \tag{A-101}$$

$$= Z_{22} - Z_{21}{Z_{11}}^{-1}Z_{12} \tag{A-102}$$

$$= {Z_{22}^{*}}^{-1} \tag{A-103}$$

and hence $R^{-1}Z_{11}S_{11}Z_{11}{}^{-1} = Z^{*}_{22}$

And now $-R^{-1}Z_{11}T_{11}Z_{11}{}^{-1} = Z^{*}_{21}$

$$-R^{-1}Z_{11}T_{11}Z_{11}{}^{-1} \stackrel{?}{=} Z^{*}_{21} \tag{A-104}$$

$$R^{-1}Z_{11}T_{11}Z_{11}{}^{-1} \stackrel{?}{=} Z^{*}_{22}Z_{21}Z_{11}{}^{-1} \tag{A-105}$$

$$R^{-1}Z_{11}T_{11} \stackrel{?}{=} Z^{*}_{22}Z_{21} \tag{A-106}$$

$$R^{-1}Z_{11}Q_{11}{}^{-1}Z_{21} \stackrel{?}{=} Z^{*}_{22}Z_{21} \tag{A-107}$$

$$\left(R^{-1}Z_{11}Q_{11}{}^{-1} - Z^{*}_{22}\right)Z_{21} \stackrel{?}{=} 0 \tag{A-108}$$

Note that $Q_{11}Z_{11}{}^{-1}R = Q_{11}Z_{11}{}^{-1}\left(Z_{11}Q_{11}^{-1}Z_{22} - Z_{21}Q_{11}{}^{-1}Z_{12}\right) = Z_{22} - Q_{11}Z_{11}{}^{-1}Z_{21}Q_{11}{}^{-1}Z_{12}$ which is $Z^{*}_{22}{}^{-1}$ from the previous equality above. Hence

$$\left(R^{-1}Z_{11}Q_{11}{}^{-1} - Z^{*}_{22}\right)Z_{21} \stackrel{?}{=} 0 \tag{A-109}$$

$$\left(Z^{*}_{22} - Z^{*}_{22}\right)Z_{21} \stackrel{?}{=} 0 \tag{A-110}$$

which holds and thus $-R^{-1}Z_{11}T_{11}Z_{11}{}^{-1} = Z^{*}_{21}$. Hence proving the one-to-one equivalence to theorem 4.20 and completing the proof.

Posterior Densities

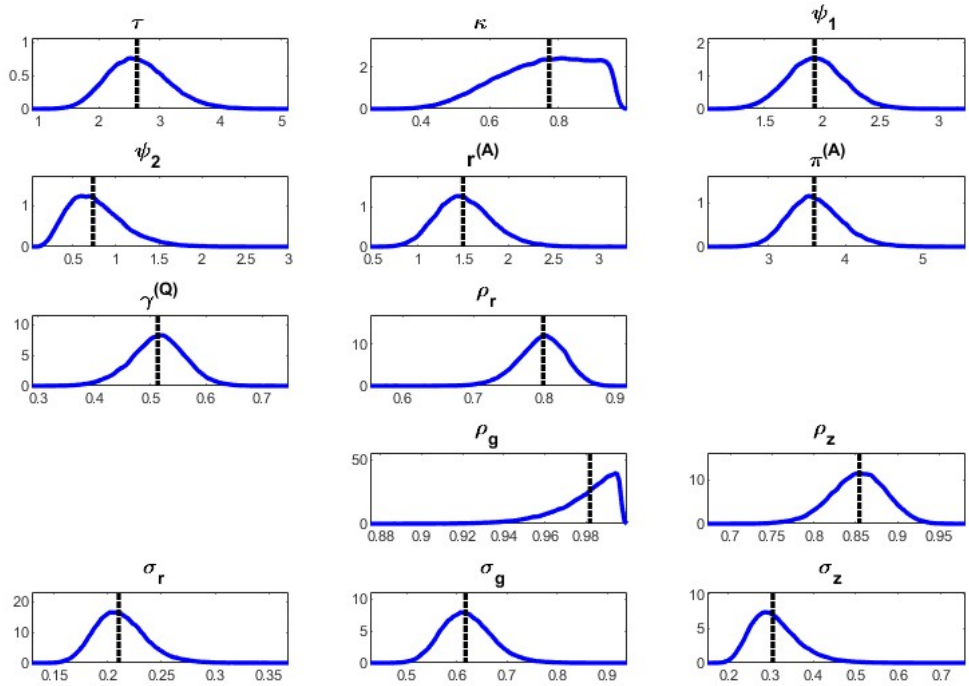


FIGURE 5. Posterior Density AR(1)

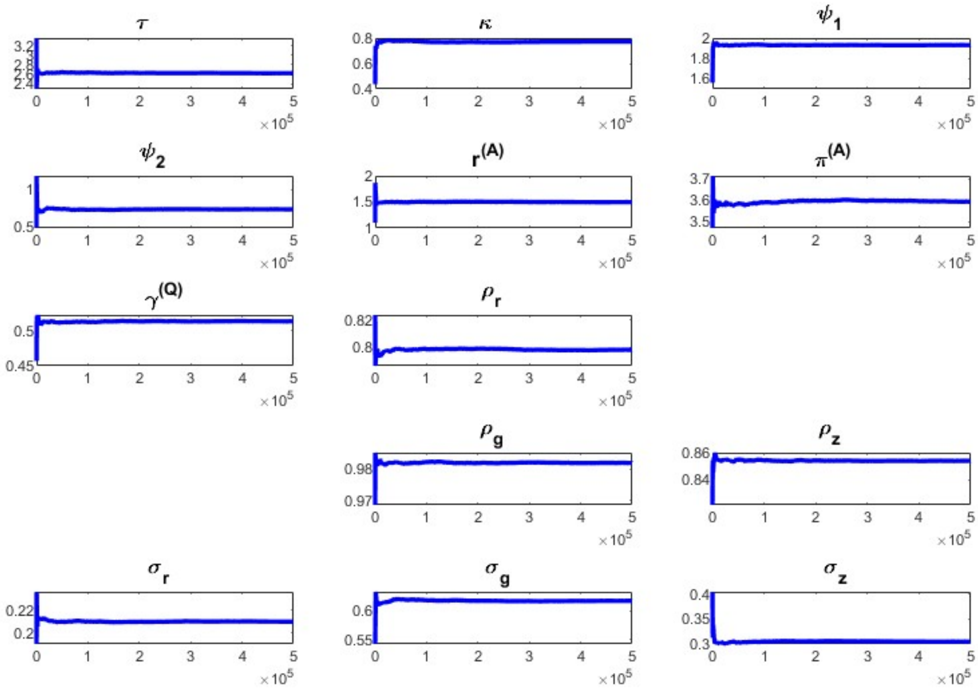


FIGURE 6. Posterior recursive Averages AR(1)

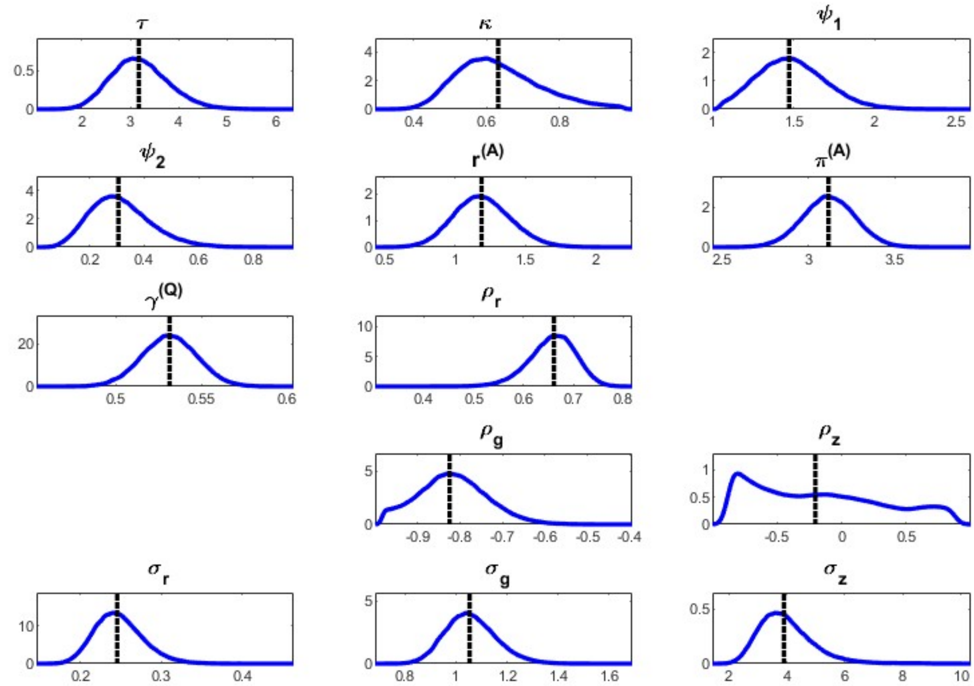


FIGURE 7. Posterior Density MA(1)

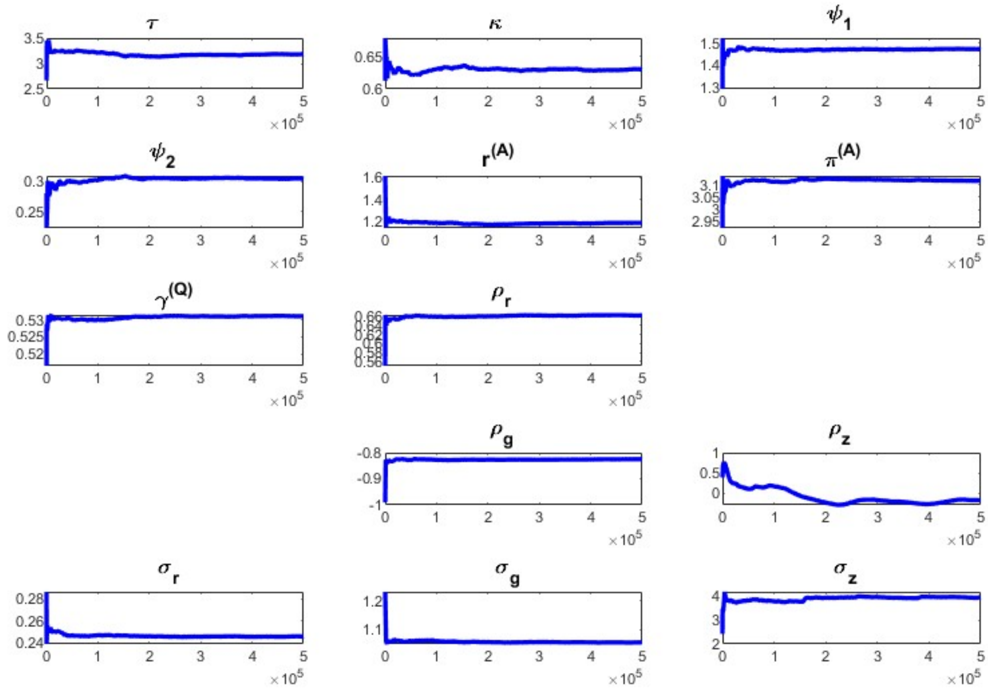


FIGURE 8. Posterior recursive Averages MA(1)

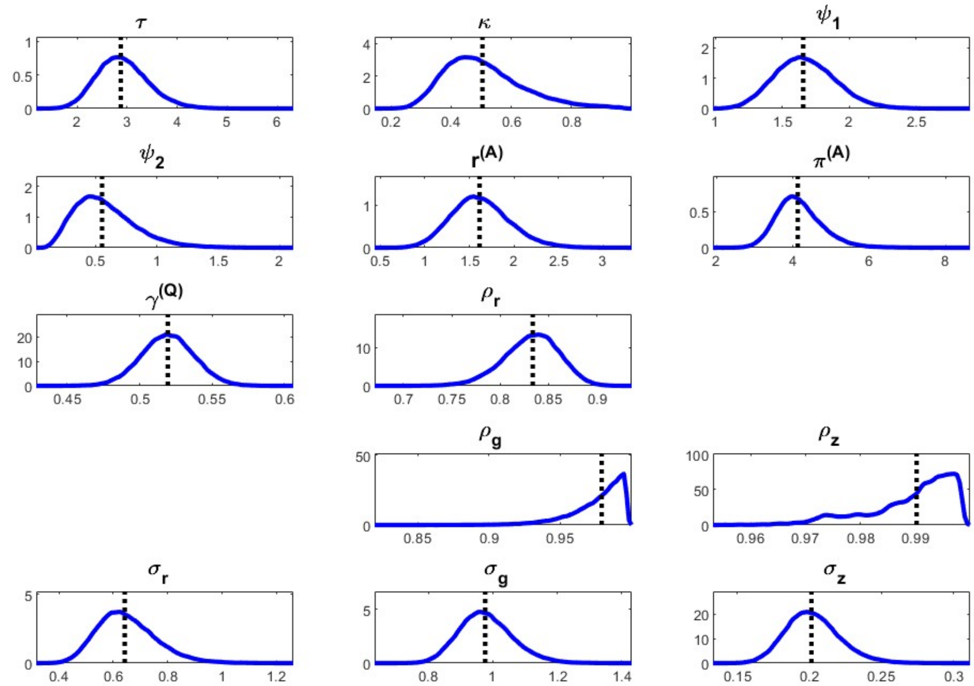


FIGURE 9. Posterior Density Log Lag

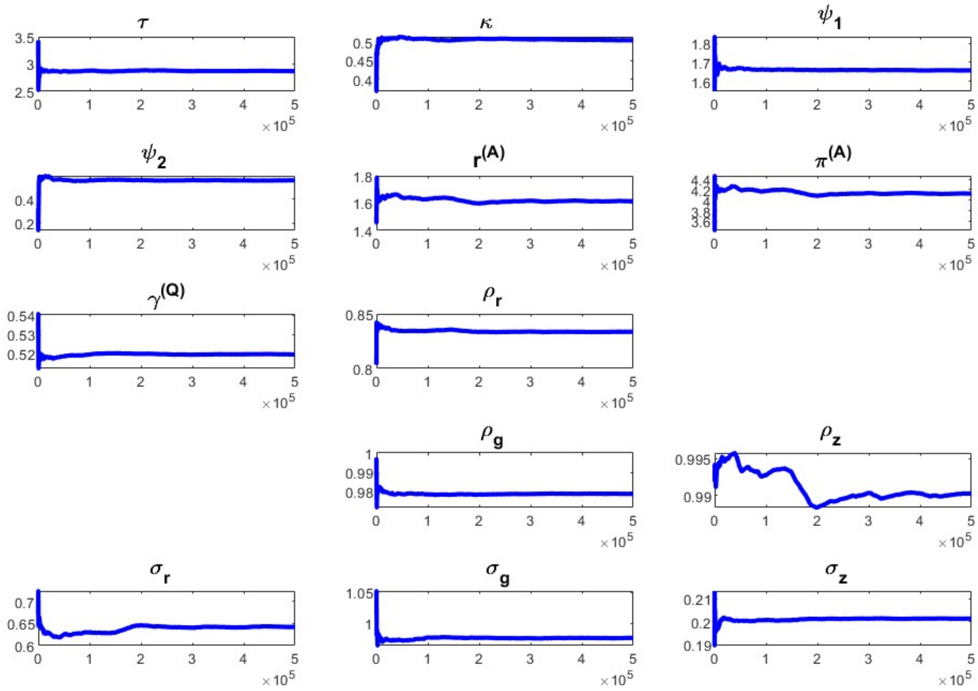


FIGURE 10. Posterior recursive Averages Log Lag

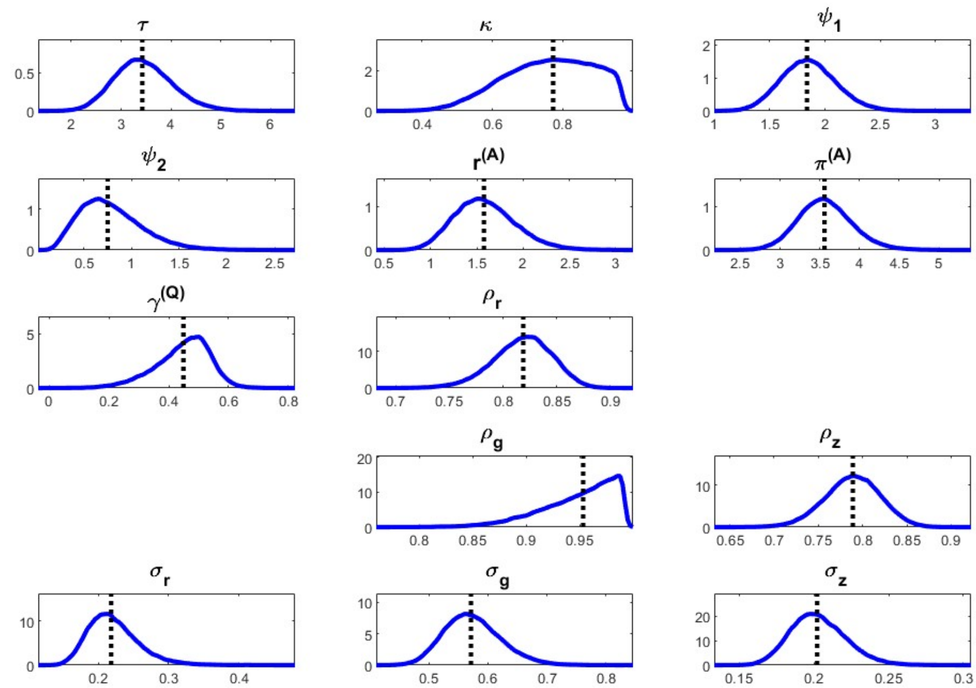


FIGURE 11. Posterior Density Log Harmonic Lag

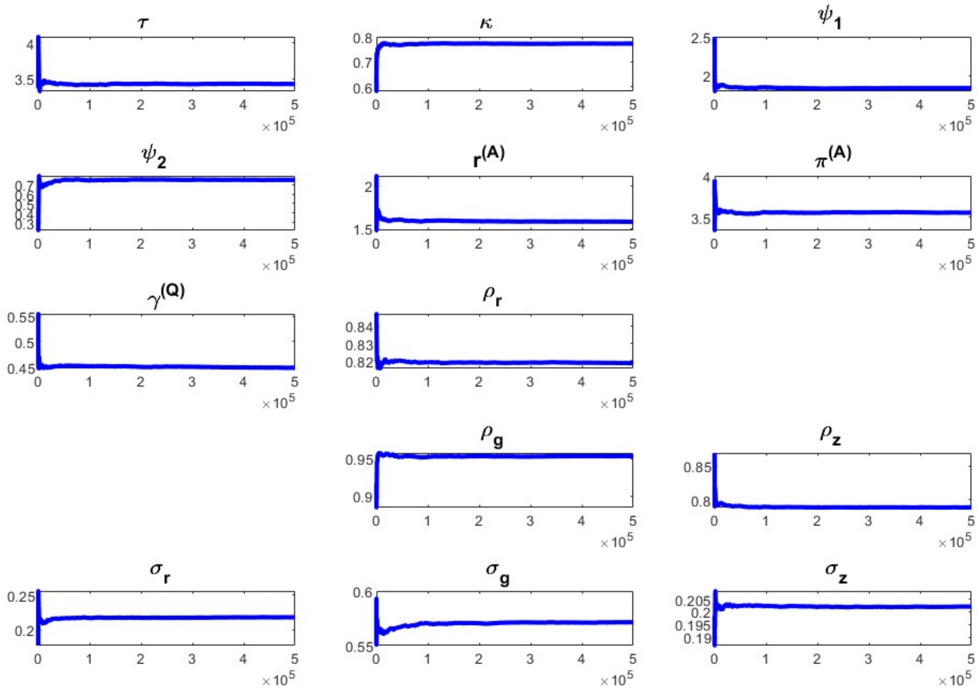


FIGURE 12. Posterior recursive Averages Log Harmonic Lag

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