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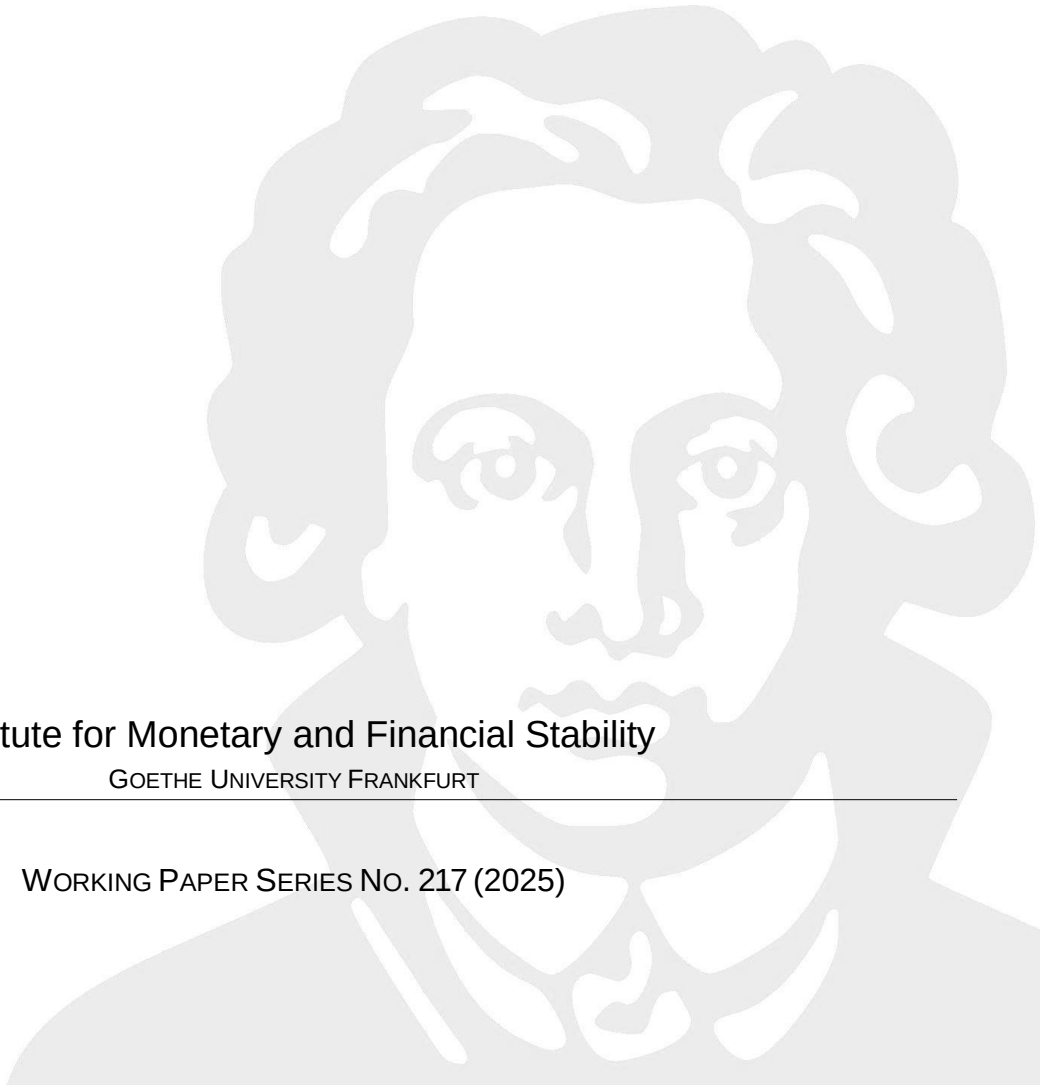

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ITERATIVE REFINEMENT OF THE QZ DECOMPOSITION FOR SOLVING LINEAR DSGE MODELS

JOHANNES HUBER AND ALEXANDER MEYER-GOHDE

ABSTRACT. The standard approach to solving linear DSGE models is to apply the QZ method. It is a one-shot algorithm that leaves the researcher with little alternative than to seek a different algorithm should the result be numerically unsatisfactory. We develop an iterative implementation of QZ that delivers the standard result as its first iteration and further refinements at each subsequent iteration. We demonstrate that our algorithm successful corrects for accuracy losses identified in particular cases of a macro finance model and does not erroneously attempt to refine sufficiently accurate solutions. *JEL classification codes*: C61, C63, E17

Keywords: Numerical accuracy; DSGE; Solution methods

1. INTRODUCTION

The major computational hurdle in the solution of linear dynamic stochastic general equilibrium (DSGE) models is the solution of the associated matrix quadratic equation - the current standard in the literature is to use a generalized Schur or QZ decomposition (Moler and Stewart, 1973; Golub and van Loan, 2013) to solve this equation.[1] Implementations of this algorithm has been provided to the profession as a one-shot tool, requiring researchers to take it or leave it. This is problematic if the accuracy of the QZ solution is perhaps lacking. We extend the methodology of solving linear DSGE models with QZ or generalized Schur decompositions by providing an iterative version that allows users to improve on a QZ solution with QZ itself.

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[1]See Dynare (Adjemian, Bastani, Juillard, Mihoubi, Perendia, Ratto, and Villemot, 2011) (henceforth Dynare) and especially Villemot (2011) for Dynare's specific implementation of QZ.

We provide an eigenvalue preserving transformation that reinterprets the companion linearization of the matrix quadratic as a subspace problem in increments of the solution or solvent. Beginning at the zero matrix, the first iteration reproduces the standard QZ approach. Further iterations refine the solution - while theoretically refinements should not be possible, this is numerically not the case: as Tisseur (2000) shows for the quadratic eigenvalue problem, QZ's backward stability is broken by the block structure that is necessary to maintain the mapping to the underlying matrix quadratic problem. We assess our iterative QZ implementation in a sets of experiments using the model of Jermann (1998) that has numerically unstable parameterization nearby to the standard parameterization. We find that our iterative QZ method successfully terminates when the solution is accurate enough, as measured by Meyer-Gohde's (2022) practical forward error bounds, and overcomes numerically inaccuracies by running a few iterations through improvement increments.

2. Iterative QZ

The linear, or first-order, approximation of a DSGE model at the steady state can typically be stated as

$$0 = AE_t[y_{t+1}] + By_t + Cy_{t-1} + D\varepsilon_t \tag{1}$$

where the coefficient matrices A, B and C are $n_y \times n_y$ and D is $n_y \times n_e$, the endogenous variables $y_t \in \mathbb{R}^{n_y}$ are a vector of size n_y ; and the vector of n_e exogenous shocks are contained in $\varepsilon_t \in \mathbb{R}^{n_e}$, where n_y and n_e are positive integers ($n_y, n_e \in \mathbb{N}$) and ε_t has a known mean zero distribution.

The solution to the linearized model (1) is a linear solution in the form,

$$y_t = P\, y_{t-1} + Q\, \varepsilon_t \tag{2}$$

a recursive solution for y_t as a function of its own past, y_{t-1}, and shocks, ε_t.

Using our linear solution (2), it yields through substitution into (1) - and recognizing that the expectation $E_t[\varepsilon_{t+1}] = 0$ is known - the following two equations

$$0 = AP^2 + BP + C, \quad 0 = (AP + B)Q + D \tag{3}$$

With the former solution being quadratic with potentially multiple solutions, a selection criteria has to be used and generally a unique (semi) stable solution P is sought by the literature, that is a P with all its eigenvalues inside the open unit circle. As Lan and

Meyer-Gohde (2014) prove the latter can be uniquely solved for Q if such a P can be found, our focus will be the former, the unilateral quadratic matrix equations (UQME).

2.1. Standard QZ

To enable our interative QZ approach, we will first express the UQME in (3) as a subspace problem by forming the first companion linearization of the matrix quadratic problem (Hammarling, Munro, and Tisseur, 2013)

$$\mathscr{A}\mathscr{X} = \mathscr{B}\mathscr{X}\mathscr{M} \tag{4}$$

with

$$\mathscr{X} = \begin{bmatrix} I \\ P \end{bmatrix} \qquad \mathscr{A} = \begin{bmatrix} C & B \\ 0 & I \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 0 & -A \\ I & 0 \end{bmatrix}, \qquad \mathscr{M} = P.$$

Clearly, any P satisfying (3) is a solution of (4). Further note that the eigenvalues of $\mathscr{M}$ are a subset of the generalized eigenvalues of the matrix pencil $\mathscr{A} - \lambda\mathscr{B}$, i.e., $\text{eig}(\mathscr{M}) \subset \text{eig}(\mathscr{A}_0, \mathscr{B}_0)$.

We will assume the conditions for the existence of the unique solvent P are fulfilled, i.e., Blanchard and Kahn's (1980) celebrated rank and order conditions, and hence that there exist a unique solvent P satisfying the UQME in (3) with eigenvalues on or inside the unit circle and all remaining eigenvalues associated with the UQME (or latent roots of the UQME, see Meyer-Gohde (2022)) are outside the unit circle.

The problem in (4) is numerically an eigenvalue problem and can thus be solved using the QZ or generalized Schur decompostion of Moler and Stewart (1973). We will derive the solution by working directly with the linear algebraic problem instead of the dynamic model as is usually done to keep the notation simple. The decomposition provides unitary Q and Z and upper triangular S and T with $Q^*\mathscr{B}Z = S$ and $Q^*\mathscr{A}Z = T$, where * indicates the complex conjugation of Q that delivers its inverse by virtue of it being a unitary matrix and where the eigenvalues of the matrix pencil $P_{\mathscr{B}\mathscr{A}}(z) = \mathscr{B}z - \mathscr{A}$, $\rho(P_{\mathscr{B}\mathscr{A}}) = \rho(P_{ST}) = \{t_{ii}/s_{ii}, \text{ if } s_{ii} \neq 0; \infty, \text{ if } s_{ii} = 0; \emptyset, \text{ if } s_{ii} = t_{ii} = 0; i = 1, \ldots, 2n_y\}$,[2] can be ordered arbitrarily to form

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} W^s \\ W^u \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} W^s \\ W^u \end{bmatrix} P \tag{5}$$

[2] If both $s_{ii} = t_{ii} = 0$ for some i, then any $z \in \mathbb{C}$ will satisfy $|\mathscr{B}z - \mathscr{A}| = 0$. This is the mundane singularity of King and Watson (1998) that reflects a poorly specified model, e.g., a repeated equation, and the definition in the text can be interpreted to ruling out such misspecifications by limiting the analysis to regular pencils.

where $Z^* \begin{bmatrix} I & P' \end{bmatrix}' = \begin{bmatrix} W^{s\prime} & W^{u\prime} \end{bmatrix}'$. We assume the decomposition is ordered so that the unstable eigenvalues are in the lower right blocks of S and T (hence S_{22} and T_{22}), wherefor the lower block equation gives

$$T_{22}W^u = S_{22}W^uP \Rightarrow W^u = T_{22}^{-1}S_{22}W^uP. \tag{6}$$

Consequently, the eigenvalues of $T_{22}^{-1}S_{22}$ are inside the unit circle and P is a (semi) stable solvent, so that we can iterate

$$W^u = T_{22}^{-1}S_{22}W^uP = \left(T_{22}^{-1}S_{22}\right)^2 W^uP^2 = \left(T_{22}^{-1}S_{22}\right)^j W^uP^j \underset{j\to\infty}{\to} 0 \tag{7}$$

for a unit root stable P. Using the definition of W^u we receive

$$0 = W^u = Z_{21}^* + Z_{22}^*P \Rightarrow P = -{Z_{22}^*}^{-1}Z_{21}^* = Z_{21}Z_{11}^{-1} \tag{8}$$

The equivalence ${Z_{22}^*}^{-1}Z_{21}^* = -Z_{21}Z_{11}^{-1}$ follows from the properties of unitary matrices and $Z_{21}Z_{11}^{-1} = Q_{11}S_{11}^{-1}T_{11}Q_{11}^{-1}$ from the first block rows of $\mathscr{A}$ and $\mathscr{B}$ in (4) and the upper triangularity of S and T. From $Q_{11}S_{11}^{-1}T_{11}Q_{11}^{-1}$, it follows that the recursion in P is indeed stable from the ordering of the eigenvalues above, i.e. the eigenvalues of the upper left block of the generalized Schur decomposition, $det(S_{11}\lambda - T_{11}) = 0$, are inside the unit circle. So the QZ decomposition applied to our matric pencil will recover the unique (semi) stable solvent P if it exists consistent with our assumption its existence.

2.2. Iterative QZ

While the QZ decomposition analytically recovers the unique solvent, numerically it is subject to finite precision. Hence, the decomposition delivers $\widehat{P} = P + \mathscr{O}(\epsilon)$ which differs from the true solution P via numerical errors $\mathscr{O}(\epsilon)$. If the solution $\widehat{P}$ is unsatisfactory, that is, if the errors, $\mathscr{O}(\epsilon)$, are too large, we would like an alternative to abandoning the QZ solution and having to find an alternate solution. That is, we would like to improve on $\widehat{P}$ using QZ. To this end we will derive an interative QZ approach that takes this approximate solution as the initialization or output from the previous iteration P_{k-1}. Using this P_{k-1}, we define a matrix to transform the original, one-shot problem

$$U_k \equiv \begin{bmatrix} I & 0 \\ P_{k-1} & I \end{bmatrix}, \qquad U_k^{-1} \equiv \begin{bmatrix} I & 0 \\ -P_{k-1} & I \end{bmatrix} \tag{9}$$

We note that U_k is an eigenvalue preserving transform[3] with

$$\mathscr{A}_k \equiv \mathscr{A} U_k, \qquad \mathscr{B}_k \equiv \mathscr{B} U_k \tag{10}$$

giving

$$\mathscr{A}\mathscr{X} = \mathscr{B}\mathscr{X}\mathscr{M} \Rightarrow \mathscr{A}U_k U_k^{-1}\mathscr{X} = \mathscr{B}U_k U_k^{-1}\mathscr{X}\mathscr{M} \Rightarrow \mathscr{A}_k\hat{\mathscr{X}}_k = \mathscr{B}_k\hat{\mathscr{X}}_k\mathscr{M}. \tag{11}$$

where $\hat{\mathscr{X}}_k$ defines the potential improvement that is key to the iterative nature of our approach

$$\hat{\mathscr{X}}_k \equiv U_k^{-1}\mathscr{X} = \begin{bmatrix} I & 0 \\ -P_{k-1} & I \end{bmatrix}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P - P_{k-1} \end{bmatrix} \tag{12}$$

and thus (with $\Delta P_k \equiv P - P_{k-1}$)

$$\mathscr{A}_k \begin{bmatrix} I \\ \Delta P_k \end{bmatrix} = \mathscr{B}_k \begin{bmatrix} I \\ \Delta P_k \end{bmatrix}\mathscr{M} \tag{13}$$

and

$$\mathscr{A}_k = \begin{bmatrix} C + BP_{k-1} & B \\ P_{k-1} & I \end{bmatrix}, \qquad \mathscr{B}_k = \begin{bmatrix} -AP_{k-1} & -A \\ I & 0 \end{bmatrix} \tag{14}$$

As above with the pencil $\mathscr{A}$, $\mathscr{B}$ we can apply the QZ decomposition to the pencil defined by $\mathscr{A}_k$, $\mathscr{B}_k$. The decomposition provides unitary Q_k and Z_k and upper triangular S_k and T_k with $Q_k^*\mathscr{B}_k Z_k = S_k$ and $Q_k^*\mathscr{A}_k Z_k = T_k$, where the eigenvalues of the matrix pencil $P_{\mathscr{B}_k\mathscr{A}_k}(z) = \mathscr{B}_k z - \mathscr{A}_k$, $\rho(P_{\mathscr{B}_k\mathscr{A}_k}) = \rho(P_{S_kT_k}) = \{t_{k,ii}/s_{k,ii}$, if $s_{k,ii} \neq 0; \infty$, if $s_{k,ii} = 0; \emptyset$, if $s_{k,ii} = t_{k,ii} = 0;\ i = 1,\ldots,2n_y\}$. From the eigenvalue preserving transformation behind $\mathscr{A}_k$ and $\mathscr{B}_k$, follows that $\rho(P_{\mathscr{B}_k\mathscr{A}_k}) = \rho(P_{\mathscr{B}\mathscr{A}})$. And analogous to (5) we receive

$$\begin{bmatrix} T_{k,11} & T_{k,12} \\ 0 & T_{k,22} \end{bmatrix}\begin{bmatrix} W_k^s \\ W_k^u \end{bmatrix} = \begin{bmatrix} S_{k,11} & S_{k,12} \\ 0 & S_{k,22} \end{bmatrix}\begin{bmatrix} W_k^s \\ W_k^u \end{bmatrix} P \tag{15}$$

where $Z_k^*\begin{bmatrix} I & \Delta P_k' \end{bmatrix}' = \begin{bmatrix} W_k^{s\prime} & W_k^{u\prime} \end{bmatrix}'$. We assume the decomposition is ordered so that the unstable eigenvalues are in the lower right blocks of S_k and T_k (hence $S_{k,22}$ and $T_{k,22}$), wherefor the lower block equation gives

$$T_{k,22}W_k^u = S_{k,22}W_k^u P \Rightarrow W_k^u = T_{k,22}^{-1}S_{k,22}W_k^u P. \tag{16}$$

[3] See the appendix for details.

Since the eigenvalues of $T_{k,22}^{-1}S_{k,22}$ are inside the unit circle and P is the (semi) stable (but unknown) solvent or $\mathscr{M}$ we can iterate

$$W_k^u = T_{k,22}^{-1}S_{k,22}W_k^u P = \left(T_{k,22}^{-1}S_{k,22}\right)^2 W_k^u P^2 = \left(T_{k,22}^{-1}S_{k,22}\right)^j W_k^u P^j \underset{j\to\infty}{\to} 0. \tag{17}$$

Using the definition of W_k^u we receive

$$0 = W_k^u = Z_{k,21}^* + Z_{k,22}^*\Delta P_k \Rightarrow \Delta P_k = -{Z_{k,22}^*}^{-1}Z_{k,21}^* = Z_{k,21}Z_{k,11}^{-1}. \tag{18}$$

Due to finite precision arithmetic, the algorithm will compute $\Delta\widehat{P}_k = \Delta P_k + \mathscr{O}_k(\epsilon)$ which is only an approximation of the true potential improvement ΔP_k subject to numerical errors $\mathscr{O}_k(\epsilon)$. Therefore, the updated approximation of P yields

$$P_k = P_{k-1} + \Delta\widehat{P}_k, \tag{19}$$

We initialize with $P_{-1} = 0$, such that the P_0 recovered above is the standard QZ solution derived by Klein (2000), Sims (2001), and Uhlig (1999) and is the standard method in Dynare.[4] We thusly interpret P_k for $k > 0$ as iterative QZ improvements and define the following iterative QZ algorithm

Algorithm 1: Iterative QZ Algorithm

Given: A, B, C, and a convergence criterion ϵ

Set $P_{-1} = 0$

While *criterion*$(\Delta P_k) > \epsilon$ **do**

Define $\mathscr{A}_k = \begin{bmatrix} C + BP_{k-1} & B \\ P_{k-1} & I \end{bmatrix}$ and $\mathscr{B}_k = \begin{bmatrix} -AP_{k-1} & -A \\ I & 0 \end{bmatrix}$

Apply the QZ algorithm $Q_k^*\mathscr{B}_k Z_k = S_k$ and $Q_k^*\mathscr{A}_k Z_k = T_k$, arranged with the unstable eigenvalues are in the lower right blocks of S_k and T_k

Set $\Delta P_k = Z_{k,21}Z_{k,11}^{-1}$

Set $P_k = P_{k-1} + \Delta P_k$

Advance $k = k + 1$

end

Return: P_k

[4] See the appendix for the details of implementing the algorithm in Dynare in accordance with its classification of variables.

Ideally we would solve all problems exactly. With finite precision, the best we can hope for is to solve a problem within machine precision, ϵ. With each calculation, we invariably inherit potential rounding or floating point errors that limit the possible accuracy of our results. Furthermore, these errors are magnified in ill-conditioned problems. Accordingly, we follow Higham (2002, ch. 3) and calculate a target threshold for the solution of the matrix quadratic based on the number of floating point calculations. Adding C and AP^2+BP involves ny^2 operations and adding AP^2 and BP another ny^2, each with a potential finite precision error ε_{MP}. The inner product $x'y$ for vectors of length n_y can be bounded by $\gamma(n_y) \equiv \frac{n_y\varepsilon_{MP}}{1-n_y\varepsilon_{MP}}$ accordingly the product of B and P has floating point errors bounded by $ny^2\gamma(n_y)$ and that of A, P, and P by $\left(ny^2\gamma(n_y)\right)^2$. This gives in total $ny^2\left[\varepsilon_{MP}+\varepsilon_{MP}\gamma(n_y)\left(\varepsilon_{MP}+\varepsilon_{MP}\gamma(n_y)\right)\right]$ which following Higham (2002, Lemma 3.3) using the rules of running error analysis gives $n_y^2\left(\varepsilon_{MP}+\gamma(n_y+2)+\gamma(2*n_y+2)\right)$. For ill conditioned problems, these error can be potentiated into errors of the solution via the condition number Ψ (see Higham (2002) in general and Meyer-Gohde (2022) for the specific DSGE matrix quadratic problem). Combining, the realistic level of accuracy for a matrix quadratic problem with a given conditioning and dimension is

$$\epsilon = \Psi * n_y^2\left(\varepsilon_{MP}+\gamma(n_y+2)+\gamma(2*n_y+2)\right) \tag{20}$$

which we take as our convergence criterion ϵ.

We expect our iterative QZ algorithm to perform favorably in terms of improving the accuracy of a solution for two reasons. Firstly, the algorithm attempts to set ΔP_k to close the gap between the current initialization P_{k-1} and the solution to the matrix quadratic P. That is, it solves for ΔP_k such that $P_k = P$. Of course, it is limited like the original QZ (i.e., the first iteration of our method) by finite precision arithmetic. However, the addition of the terms P_{k-1} in the lower left of $\mathscr{A}_k$ and $-AP_{k-1}$ in the upper left of $\mathscr{B}_k$ removes two of the five zero/identity blocks in the original formulation of $\mathscr{A}$ and $\mathscr{B}$. This is significant as the breakdown of the backward stability of the QZ algorithm for solving the matrix quadratic problem as identified by Tisseur (2000) is caused by the large number of mandatory zeros and ones to maintain the mapping of the matrix quadratic to a generalized eigenvalue problem, which, as Meyer-Gohde (2022) shows, leads to QZ based algorithms in the DSGE literature being more prone to numerical instability than other, less common methods. We will now confirm that with the reduction of these mandatory zeros and ones by 40%, our iterative QZ is able to overcome numerical instability as identified in a specific macro-finance example by Meyer-Gohde (2022).

3. A Macro Finance: Application

We run through a set of experiments to assess the iterative QZ in the model of Jermann (1998). We test how our algorithm performs under both conditions when the standard QZ works well and when it does not - namely Meyer-Gohde's (2022) parameterizations and formulations of Jermann's (1998) macro finance model. This model is attractive as it successfully replicates key finance variables, such as the average equity premium, in a production based asset pricing framework - that is, with endogenous production and consumption in a compact parsimonious DSGE model. Furthermore, Heiberger, Klarl, and Maußner (2017) analyze a similar model, with habit formation in labor along with consumption, for the numerical accuracy of linear DSGE solutions.

Jermann's (1998) model of habit formation and adjustment costs gives agents a strong motive to care about volatile consumption streams but inhibits their ability to respond which drive the success in matching the macro and finance moments. The habit stock, X_t, is internal as households internalize the effect of consumption today on the habit they will face tomorrow, yielding marginal utility from consumption, λ_t, in the pricing kernel $m_{t+1} \equiv \beta\lambda_{t+1}/\lambda_t$ of

$$\lambda_t \equiv \partial\left(u\left(c_t, X_t\right) + \beta E_t\left[u\left(c_{t+1}, X_{t+1}\right)\right]\right)/\partial c_t \tag{21}$$

If habit formation is external, this is simply $u_c(c_t, X_t)$, here with the habit internalized it is a function of the previous period's consumption, $X_t \equiv X(c_{t-1})$, and is given by

$$\lambda_t = u_c(c_t, X(c_{t-1})) + \beta E_t[u_X(c_{t+1}, X(c_t))X_c(c_t)]$$

We assume $u(\cdot_t) = (\cdot_t)^{1-\tau}/(1-\tau)$ with curvature τ and habit $\cdot_t = c_t - bc_{t-1}$ with the degree of habit formation b. Households' ability to reduce the volatility of consumption streams as desired is impeded by adjustment costs in capital accumulation

$$k_t = (1 - \delta/k_{t-1})k_{t-1}$$

where the capital adjustment cost function is $\phi(i_t/k_{t-1}) = \frac{\tilde{b}}{1-\xi}(i_t/k_{t-1})^{1-\xi} + \tilde{c}$ with $\tilde{b}$ and $\tilde{c}$ set such that the steady state matches the case without adjustment costs and $1/\xi$ is the elasticity of the investment-capital ratio with respect to Tobin's q.

We examine two parameterizations that can be found in table 1. The baseline calibration is the calibration given by Jermann (1998) slightly adjusted to match the his empirical moments, the average risk premium, the risk free rate and the standard deviations of output, investment and consumption growth, in the most accurate linear method exactly.

TABLE 1. Calibrations for Jermann's (1998) DSGE Macro-Finance Model

	τ	b	$\beta^* = \beta\overline{a}^{1-\tau}$	ξ	σ_Z	
Baseline	7.92	0.74	0.99	3.75	0.999%	
Alternative	29.96	0.66	0.94	0.76	0.997%	
	α	$\overline{a}$	δ	ρ_Z	N_{ss}	SS_{adjc}
Common	0.36	0.005	0.025	0.99	1	1

The alternative parameterization is merely a near-by calibration with an increase in the curvature of the utility kernel - increasing households' unconditional sensitivity to volatile consumption streams - partially attenuated by a reduction in degree of habit formation but an significant increase in the elasticity in the adjustment costs - making them, however, more able to respond to this volatility. Just as different methods that are theoretically identical can have different numerical consequences, so too can different formulations of a model that are theoretically identical have different numerical consequences. Consider the application of the Lucas asset pricing equation to pricing economic capital

$$1 = E_t\left[e^{\hat{m}_{t+1}+\hat{R}_{t+1}}\right] \tag{22}$$

or equivalently

$$1 = \beta E_t\left[e^{-\tau\overline{a}+\hat{R}_{t+1}}\frac{\mu_{t+1}}{\mu_t}\right], \text{ as } e^{\hat{m}_t} = \beta e^{-\tau\overline{a}}\frac{\mu_t}{\mu_{t-1}} \tag{23}$$

and μ_t is deterministically detrended λ_t from (21) above. In the results above, the formulation (23) was used, now we replace (23) with (22). This theoretically makes no difference, but numerically changes the problem significantly (Meyer-Gohde, 2022). To see this intuitively, notice that previously, there were entries in the B matrix (associated with contemporaneous variables) as well as in the A matrix (associated with future variables) in the row associated with this equation. Now with (22), there are only entries in the A matrix (associated with future variables) in the row associated with this equation.

Table 2 contains the results for both parameterizations and the reformulation of the Euler equation. The data row repeats the moments in post-war US data reported by Jermann (1998) in his study. The first set of results reports the baseline calibration and model definition. We see that our iterative QZ algorithm terminates after the initial iteration and produces the same results as Dynare's QZ and the best (in terms of forward error which is Dynare's cyclic reduction (CR) method) performing algorithm for this constellation of Jermann's (1998) model as found by Meyer-Gohde (2022).[5] Thus, the

[5]Note that the forward errors for our iterative QZ and Dynare's QZ differ by roughly an order of magnitude. For the experiments with Jermann's (1998) model, we implement the naïve version of our

	$E[rp]$	$E\left[R_t^f\right]$	$std(\log y_t)$	$std(\log c_t)$	Cond. Num.	Forward Error
Data	6.18	0.8	0.01	0.51	–	–
Baseline Calibration						
Iterative QZ (1 iteration)	6.18	0.8	0.01	0.51	9.4e+05	1.12e-12
Dynare QZ	6.18	0.8	0.01	0.51	9.4e+05	2.06e-13
In Meyer-Gohde (2022) (Best: Dynare CR)	6.18	0.8	0.01	0.51	9.4e+05	2.23e-15
Alternative Calibration						
Iterative QZ (2 iterations)	6.23	0.824	0.01	0.519	1.06e+05	3.07e-08
Dynare QZ	6.09	0.696	0.01	0.584	1.22e+05	1.83e-01
In Meyer-Gohde (2022) (Best: Binder-Peseran)	6.23	0.824	0.01	0.519	1.06e+05	1.77e-15
Alternative Calibration, Alternate Equations						
Iterative QZ (8 iterations)	6.23	0.824	0.01	0.519	1.05e+05	7.19e-06
Dynare QZ	6.29	-0.326	0.01	0.531	7.65e+04	1.16e+01
In Meyer-Gohde (2022) (Best: AIM)	6.23	0.824	0.01	0.519	1.05e+05	1.72e-15

TABLE 2. Jermann (1998) Moments and Errors

expected level of accuracy is returned by the standard QZ algorithm and the macro finance results in terms of moments of economic variables are identical in the QZ and the most accurate algorithm. The next set of results is for the alternative calibration with higher utility curvature counterbalanced by a lower degree of habit formation and higher elasticity of adjustment costs. Our iterative QZ algorithm does *not* terminate after the initial iteration and chooses instead to run a second iteration. Compared with the standard one-shot implementation of QZ there is a gain of forward error accuracy of

algorithm as laid out above. Dynare's implementation reduces the dimensionality of the QZ problem by grouping variables and structuring the matrix quadratic according to the classification of "static", "purely forward", "purely backward looking", and "mixed" variables, see Villemot (2011). The analogous dimension reduction for our iterative QZ algorithm is detailed in the appendix. We choose to report the naïve results here, particularly for the baseline calibration, to highlight the importance of the actual numerical implementation of QZ that leads to a one order of magnitude improvement in the forward error.

roughly seven degrees of magnitude and our iterative QZ successfully recovers the macro finance moments produced by the most accurate method reported in Meyer-Gohde (2022). The last set of results with both the alternative calibration and formulation of the Euler equations presents a formidable challenge - many methods in Meyer-Gohde (2022) failed to produce a solution at all and Dynare's QZ even predicts a *negative* risk-free rate for this case. With 8 iterations, our iterative QZ has to put in extra effort with this particular problem to achieve the target forward error. Upon convergence, the algorithm again is roughly seven degrees of magnitude more accurate than Dynare's QZ and produces moments that match those of the most accurate method for this constellation. Thus, for the model of Jermann (1998), we conclude that our method successfully identifies situations where the standard QZ method is sufficient and rectifies those that are not.

In sum, our experiments clearly demonstrate the utility and reliability of our algorithm - iterating over the standard QZ results when they are not sufficiently accurate and refraining from iterating when the are. The only additional costs over the standard implementation of QZ are the calculations of conditioning numbers and forward errors for each model, which is standard numerical practice to ensure numerical stability.

4. Conclusion

We have introduced and developed an iterative implementation of the QZ algorithm for solving linear DSGE models as alternatives to the one-shot QZ methods in the literature. The first iteration initialized on the zero matrix returns the standard QZ output, further iterations provide improvement increments and we continue until the solution achieves the accuracy permitted by Meyer-Gohde's (2022) forward error bounds.

In a set of experiments using Jermann's (1998) macro finance model, we find that our proposed algorithm returns the standard QZ output should it already be accurate enough and otherwise runs a few iterations until this accuracy is achieved. Thus we have provided an entirely QZ based method to improve the accuracy of solutions of linear DSGE models alongside the alternative iterative methods developed in Meyer-Gohde and Saecker (2022), Meyer-Gohde (2023), Binder and Meyer-Gohde (2024), and Huber, Meyer-Gohde, and Saecker (2023). As these studies already demonstrated, there is likely no single one-size-fits-all algorithm and our iterative QZ based algorithm should be considered a complement of the different methods for solving linear DSGE models.

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APPENDIX

4.1. Eigenvalue Preservation

The eigenvalue preservation can be seen by comparison. The eigenvalues of the pencil defined by $\mathscr{A}$, $\mathscr{B}$ are given by $\lambda \in \mathbb{C}$ such that

$$0 = \det(\mathscr{A} - \lambda\mathscr{B}) \Rightarrow 0 = \det\left(A\lambda^2 + B\lambda + C\right) \tag{A1}$$

The eigenvalues of the pencil defined by $\mathscr{A}_k$, $\mathscr{B}_k$ are given by $\mu \in \mathbb{C}$ such that

$$\begin{aligned}
0 &= \det\left(\mathscr{A}_k - \mu\mathscr{B}_k\right) && \text{(A2)}\\
&= \det\left(\begin{bmatrix} C + BP_{k-1} & B \\ P_{k-1} & I \end{bmatrix} - \mu \begin{bmatrix} -AP_{k-1} & -A \\ I & 0 \end{bmatrix}\right) && \text{(A3)}\\
&= \det\left(\begin{bmatrix} AP_{k-1}\mu + C + BP_{k-1} & A\mu + B \\ -I\mu + P_{k-1} & I \end{bmatrix}\right) && \text{(A4)}\\
& && \text{(A5)}
\end{aligned}$$

As $-I\mu + P_{k-1}$ and I commute

$$\begin{aligned}
0 &= \det\left(\mathscr{A}_k - \mu\mathscr{B}_k\right) && \text{(A6)}\\
&= \det\left[\left(AP_{k-1}\mu + C + BP_{k-1}\right)I - \left(A\mu + B\right)\left(P_{k-1} - I\mu\right)\right] && \text{(A7)}\\
&= \det\left[AP_{k-1}\mu + C + BP_{k-1} - A\mu P_{k-1} - BP_{k-1} + A\mu^2 + B\mu\right] && \text{(A8)}\\
&= \det\left(C + A\mu^2 + B\mu\right) && \text{(A9)}
\end{aligned}$$

Inspection shows that λ and μ are identical.

4.2. Detailed Dynare Topology

Here we summarize the details in the matrix quadratic that follows from the typology of variables from Dynare as laid out in Villemot (2011). See Meyer-Gohde and Saecker (2022) for additional details.

Subdividing the system of equations in accordance with the QR decomposition yields

$$\underbrace{\begin{array}{c|ccc} & n^s & n^{--} & n^m \quad n^{++} \\ n^s & 0 & 0 & \check{\mathbf{A}}^+_{n^s\times n^+} \\ n^{--} & 0 & 0 & \\ n^m & 0 & 0 & \tilde{\mathbf{A}}^+_{n^d\times n^+} \\ n^{++} & 0 & 0 & \end{array}}_{A_{n\times n}} \mathbf{P}^2_{n\times n} + \underbrace{\begin{array}{c|ccc} & n^s & n^{--} & n^m \quad n^{++} \\ n^s & \check{\mathbf{A}}^{0s} & & \check{\mathbf{A}}^{0d}_{n^s\times n^d} \\ n^{--} & 0 & & \\ n^m & 0 & & \tilde{\mathbf{A}}^0_{n^d\times n^d} \\ n^{++} & 0 & & \end{array}}_{B_{n\times n}} \mathbf{P}_{n\times n} + \underbrace{\begin{array}{c|ccc} & n^s & n^{--} & n^m \quad n^{++} \\ n^s & 0 & \check{\mathbf{A}}^-_{n^s\times n^-} & 0 \\ n^{--} & 0 & & 0 \\ n^m & 0 & \tilde{\mathbf{A}}^-_{n^d\times n^-} & 0 \\ n^{++} & 0 & & 0 \end{array}}_{C_{n\times n}} = \mathbf{0}_{n\times n}$$

where n^d is the number of dynamic variables, the sum of number of purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. The number of forward-looking variables, n^+, is the sum of the number of mixed, n^m, and purely forward-looking variables, n^{++}, and the number of backward-looking variables, n^-, is the sum of the number of purely backward-looking, n^{--} and mixed variables n^m. Hence, the number of endogenous variables is the sum of the number of static, n^s, and dynamic variables, n^d, or

the sum of the number of static, n^s, purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. The dimensions satisfy the following

$$n^d = n^{--} + n^m + n^{++}, \quad n^+ = n^m + n^{++}, \quad n^- = n^{--} + n^m, \quad n = n^s + n^d = n^s + n^{--} + n^m + n^{++}$$

The transition matrix, P, from (2) that solves the matrix equation (3) can be subdivided in accordance to Dynare's typology as

$$\mathbf{P} = \begin{array}{c} \\ n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} \begin{array}{c} \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \left[\begin{array}{cccc} \mathbf{P}_{s,s} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{P}_{s,++} \\ \mathbf{P}_{--,s} & \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \mathbf{P}_{m,s} & \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \mathbf{P}_{++,s} & \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array} = \begin{array}{c} n \\ \left[\begin{array}{cccc} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++} \end{array}\right] \end{array} = \begin{array}{c} n \\ \left[\begin{array}{c} \mathbf{P}_{s,\bullet} \\ \mathbf{P}_{--,\bullet} \\ \mathbf{P}_{m,\bullet} \\ \mathbf{P}_{++,\bullet} \end{array}\right] \end{array}$$

The matrix quadratic can be expressed as

$$\begin{aligned} \underset{n\times n}{\mathbf{M}}(\underset{n\times n}{\mathbf{P}}) &= \underset{n\times n}{\mathbf{A}}\,\mathbf{P}^2 + \underset{n\times n}{\mathbf{B}}\,\mathbf{P} + \underset{n\times n}{\mathbf{C}} \\ &= \underbrace{(\mathbf{AP}+\mathbf{B})}_{\equiv\mathbf{G}}\mathbf{P} + \mathbf{C} \end{aligned}$$

For a solvent P of the matrix quadratic, taking the structure of C from the Dynare typology above into account yields

$$\mathbf{M}(\mathbf{P}) = 0 = \mathbf{GP} + \mathbf{C}$$

$$= \mathbf{G} \begin{array}{c} n \\ \left[\begin{array}{cccc} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++} \end{array}\right] \end{array} + \begin{array}{c} \\ n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} \begin{array}{c} \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \left[\begin{array}{ccc} \mathbf{0} & \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{array}\right] \end{array}$$

Following Meyer-Gohde and Saecker (2022), who apply corollary 4.5 of Lan and Meyer-Gohde (2014), if P is the unique solvent of $M(P)$ stable with respect to the closed unit circle, $\mathbf{G}$ has full rank and hence the columns of P associated with nonzero columns in C, the static and forward-looking variables are zero $\rightarrow \mathbf{P}_{\bullet,s} = \underset{n\times n^s}{\mathbf{0}}$, $\mathbf{P}_{\bullet,++} = \underset{n\times n^{++}}{\mathbf{0}}$, whence $\mathbf{P}$ is $\mathbf{P} = \begin{array}{c} n \\ \left[\begin{array}{cccc} \mathbf{0} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{0} \end{array}\right] \end{array}$ and $\mathbf{M}(\mathbf{P}) = \left[\begin{array}{cccc} \underset{n\times n^s}{\mathbf{0}} & \underset{n\times n^{--}}{\mathbf{M}(\mathbf{P})^{--}} & \underset{n\times n^m}{\mathbf{M}(\mathbf{P})^m} & \underset{n\times n^{++}}{\mathbf{0}} \end{array}\right]$. Consequentially, the first n^s rows of the matrix quadratic, taking $\begin{array}{c} n \\ \left[\begin{array}{c} \mathbf{P}_{--,\bullet} \\ \mathbf{P}_{m,\bullet} \\ \mathbf{P}_{++,\bullet} \end{array}\right] \end{array}$ as given, yield $\begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ n^s \left[\begin{array}{cc} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array}$ as

$$\begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ n^s \left[\begin{array}{cc} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array} = -\left[\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}}\right]^{-1} \left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^m \\ n^{++} \end{array} \left[\begin{array}{cc} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array} \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^{--} \\ n^m \end{array} \left[\begin{array}{cc} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \end{array}\right] \end{array} \right.$$

$$
\left.
\begin{array}{c}
\\
+\underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}}
\end{array}
\begin{array}{c}
\begin{array}{cc} n^{--} & n^m \end{array} \\
\begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array}
\left[\begin{array}{c|c}
\mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline
\mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline
\mathbf{P}_{++,--} & \mathbf{P}_{++,m}
\end{array}\right]
\end{array}
+\underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}}
\right)
$$

and the first n^s rows of P are $\underset{n^s\times n}{\mathbf{P}_{s,\bullet}} = \begin{array}{c} \\ n^s \end{array} \begin{array}{c} \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{0} \end{array}\right] \end{array}$.

The last n^d columns and rows of P solve the reduced matrix quadratic equation

$$
\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{0} & & \\ \hline \mathbf{0} & & \\ \hline \mathbf{0} & & \end{array}\right] \end{array}
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^+}
\underbrace{\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array}}_{\underset{n^d\times n^d}{\breve{\mathbf{P}}}}
\cdot \underset{n^d\times n^d}{\tilde{\mathbf{P}}} + \underset{n^d\times n^d}{\tilde{\mathbf{A}}^0} \underset{n^d\times n^d}{\tilde{\mathbf{P}}}
$$

$$
+ \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} & & \mathbf{0} \\ \hline & \tilde{\mathbf{A}}^- & \mathbf{0} \\ \hline & & \mathbf{0} \end{array}\right] \end{array}
\underset{n^d\times n^-}{}
$$

$$
= \underset{n^d\times n^d}{\tilde{\mathbf{M}}(\tilde{\mathbf{P}})} = \begin{array}{c} \\ n^d \end{array} \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \left[\begin{array}{c|c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\mathbf{0}}
$$

Recalling that $\underset{n\times n^{++}}{\mathbf{P}_{\bullet,++}} = \mathbf{0}$, $\tilde{\mathbf{P}}$ can be reduced and four submatrices $\mathbf{P}^{--}$, $\mathbf{P}^{-}$, $\mathbf{P}^{+}$ and $\mathbf{P}^{++}$ defined via

$$
\tilde{\mathbf{P}} = \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array}
= \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{0} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{0} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{0} \end{array}\right] \end{array}
\equiv \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \underset{n^{--}\times n^-}{\mathbf{P}^{--}} & \mathbf{0} & \mathbf{0} \\ \hline \underset{n^+\times n^-}{\mathbf{P}^{+}} & \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}
\equiv \begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \underset{n^-\times n^-}{\mathbf{P}^{-}} & \mathbf{0} \\ \hline \underset{n^{++}\times n^-}{\mathbf{P}^{++}} & \mathbf{0} \end{array}\right] \end{array}
$$

where $\underset{n^{--}\times n^-}{\mathbf{P}^{--}} \equiv \begin{array}{c} \\ n^{--} \end{array} \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \end{array}\right] \end{array}$, $\underset{n^-\times n^-}{\mathbf{P}^{-}} \equiv \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^m \\ n^{--} \end{array} \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \end{array}\right] \end{array}$, $\underset{n^+\times n^-}{\mathbf{P}^{+}} \equiv \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \begin{array}{c} n^m \\ n^{++} \end{array} \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array}$,

and $\underset{n^{++}\times n^-}{\mathbf{P}^{++}} \equiv \begin{array}{c} \\ n^{++} \end{array} \begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \left[\begin{array}{c|c} \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array}$.

We can now write the matrix quadratic as

$$
\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{0} & & \\ \hline \mathbf{0} & & \\ \hline \mathbf{0} & & \end{array}\right] \end{array}
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^+}
\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \underset{n^{--}\times n^-}{\mathbf{P}^{--}} & & \mathbf{0} \\ \hline & \underset{n^+\times n^-}{\mathbf{P}^{+}} & \mathbf{0} \\ \hline & & \mathbf{0} \end{array}\right] \end{array}
\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} & & \mathbf{0} \\ \hline \underset{n^-\times n^-}{\mathbf{P}^{-}} & & \mathbf{0} \\ \hline & \underset{n^{++}\times n^-}{\mathbf{P}^{++}} & \mathbf{0} \end{array}\right] \end{array}
+ \underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}
\begin{array}{c} \begin{array}{ccc} n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{0} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{0} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{0} \end{array}\right] \end{array}
$$

$$\begin{array}{c} \\ n^{--} \\ + \; n^{m} \\ n^{++} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \left[\begin{array}{c|c|c} & \mathbf{0} \\ \tilde{\mathbf{A}}^{-}_{n^d\times n^-} & \mathbf{0} \\ & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\tilde{\mathbf{M}}(\tilde{\mathbf{P}})} = \begin{array}{c} n^d \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \left[\begin{array}{c|c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\mathbf{0}}$$

Multiplying this out and keeping only the generically nonzero columns gives

$$\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}\;\underset{n^+\times n^-}{\mathbf{P}^{+}}\;\underset{n^-\times n^-}{\mathbf{P}^{-}} + \underset{n^d\times n^+}{\tilde{\mathbf{A}}^{0+}}\;\underset{n^+\times n^-}{\mathbf{P}^{+}} + \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{0-}}\;\underset{n^-\times n^-}{\mathbf{P}^{-}} + \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}} = \underset{n^d\times n^-}{\mathbf{0}} \tag{A10}$$

where

$$\underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} = \begin{array}{c} n^d \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \left[\begin{array}{c|c|c} \tilde{\mathbf{A}}^{0-}_{n^d\times n^-} & \mathbf{0} \end{array}\right] \end{array} + \begin{array}{c} n^d \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \left[\begin{array}{c|c|c} \mathbf{0} & \tilde{\mathbf{A}}^{0+}_{n^d\times n^+} \end{array}\right] \end{array}$$

As stated by Villemot (2011), the allocation of the middle columns (the n^m ones associated with "mixed" variables) between $\tilde{\mathbf{A}}^{0-}$ and $\tilde{\mathbf{A}}^{0+}$ is not unique.

The companion linearization of (A10) analogous to the main text is

$$\begin{array}{c} \\ n^d \\ n^m \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} \tilde{\mathbf{A}}^{-}_{n^d\times n^-} & \tilde{\mathbf{A}}^{0+}_{n^d\times n^+} \\ \mathbf{0}_{n^m\times n^-} & I^{+}_{n^m\times n^+} \end{array}\right] \end{array}\!\!\begin{array}{c} n^- \\ n^+ \end{array}\!\!\left[\begin{array}{c} I_{n^-\times n^-} \\ \mathbf{P}^{+}_{n^+\times n^-} \end{array}\right] = \begin{array}{c} \\ n^d \\ n^m \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} -\tilde{\mathbf{A}}^{0-}_{n^d\times n^-} & -\tilde{\mathbf{A}}^{+}_{n^d\times n^+} \\ I^{-}_{n^m\times n^-} & \mathbf{0}_{n^m\times n^+} \end{array}\right] \end{array}\!\!\begin{array}{c} n^- \\ n^+ \end{array}\!\!\left[\begin{array}{c} I_{n^-\times n^-} \\ \mathbf{P}^{+}_{n^+\times n^-} \end{array}\right] \underset{n^-\times n^-}{\mathbf{P}^{-}} \tag{A11}$$

where $\underset{n^m\times n^+}{I^{+}} = \begin{array}{c} n^m \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^m & n^{++} \end{array} \\ \left[\begin{array}{c|c} I_{n^m\times n^m} & \mathbf{0}_{n^m\times n^{++}} \end{array}\right] \end{array}$ and $\underset{n^m\times n^-}{I^{-}} = \begin{array}{c} n^m \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^{--} & n^m \end{array} \\ \left[\begin{array}{c|c} \mathbf{0}_{n^m\times n^{--}} & I_{n^m\times n^m} \end{array}\right] \end{array}$ are selection matrices.

This is (4), note that $n^d + n^m = n^- + n^+$

$$\underset{(n^d+n^m)\times(n^-+n^+)}{\mathscr{A}}\;\underset{(n^-+n^+)\times n^-}{\mathscr{X}} = \underset{(n^d+n^m)\times(n^-+n^+)}{\mathscr{B}}\;\underset{(n^-+n^+)\times n^-}{\mathscr{X}}\;\underset{(n^-+n^+)\times n^-}{\mathscr{M}} \tag{A12}$$

now with

$$\mathscr{X} = \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} I_{n^-\times n^-} \\ \mathbf{P}^{+}_{n^+\times n^-} \end{array}\right] \end{array} \qquad \mathscr{A} = \begin{array}{c} \\ n^d \\ n^m \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} \tilde{\mathbf{A}}^{-}_{n^d\times n^-} & \tilde{\mathbf{A}}^{0+}_{n^d\times n^+} \\ \mathbf{0}_{n^m\times n^-} & I^{+}_{n^m\times n^+} \end{array}\right] \end{array}, \qquad \mathscr{B} = \begin{array}{c} \\ n^d \\ n^m \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} -\tilde{\mathbf{A}}^{0-}_{n^d\times n^-} & -\tilde{\mathbf{A}}^{+}_{n^d\times n^+} \\ I^{-}_{n^m\times n^-} & \mathbf{0}_{n^m\times n^+} \end{array}\right] \end{array}, \qquad \mathscr{M} = \underset{n^-\times n^-}{\mathbf{P}^{-}}$$

Now we can begin with an approximate solution $\underset{n^+\times n^-}{\mathbf{P}^{+}_{k-1}}$ and define the eigenvalue preserving transformation

$$U_k \equiv \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} I_{n^-\times n^-} & \mathbf{0}_{n^-\times n^+} \\ \mathbf{P}^{+}_{k-1\;n^+\times n^-} & I_{n^+\times n^+} \end{array}\right] \end{array}, \qquad U_k^{-1} \equiv \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^- & n^+ \end{array} \\ \left[\begin{array}{c|c} I_{n^-\times n^-} & \mathbf{0}_{n^-\times n^+} \\ -\mathbf{P}^{+}_{k-1\;n^+\times n^-} & I_{n^+\times n^+} \end{array}\right] \end{array} \tag{A13}$$

giving

$$\mathscr{A}_k\hat{\mathscr{X}}_k = \mathscr{B}_k\hat{\mathscr{X}}_k\mathscr{M} \tag{A14}$$

or

$$\begin{array}{c} \\ \mathscr{A}_k \end{array} \begin{array}{c} \begin{matrix} n^- \end{matrix} \\ \begin{bmatrix} I_{n^- \times n^-} \\ \Delta \mathbf{P}_k^+ \end{bmatrix} \\ \begin{matrix} n^+ \times n^- \end{matrix} \end{array} = \begin{array}{c} \\ \mathscr{B}_k \end{array} \begin{array}{c} \begin{matrix} n^- \end{matrix} \\ \begin{bmatrix} I_{n^- \times n^-} \\ \Delta \mathbf{P}_k^+ \end{bmatrix} \\ \begin{matrix} n^+ \times n^- \end{matrix} \end{array} \mathscr{M} \tag{A15}$$

as

$$\hat{\mathscr{X}}_k \equiv U_k^{-1} \mathscr{X}_k = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} I_{n^- \times n^-} & \mathbf{0}_{n^- \times n^+} \\ -\mathbf{P}_{k-1}^+ & I_{n^+ \times n^+} \\ n^+ \times n^- & \end{bmatrix} \end{array} \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} I_{n^- \times n^-} \\ \mathbf{P}_k^+ \\ n^+ \times n^- \end{bmatrix} = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} I_{n^- \times n^-} \\ \mathbf{P}_k^+ \quad -\mathbf{P}_{k-1}^+ \\ n^+ \times n^- \quad n^+ \times n^- \end{bmatrix}$$

and

$$\mathscr{A}_k \equiv \mathscr{A} U_k = \begin{array}{c} \\ n^d \\ n^m \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} \tilde{\mathbf{A}}^- + \tilde{\mathbf{A}}^{0+} \mathbf{P}_{k-1}^+ & \tilde{\mathbf{A}}^{0+} \\ n^d \times n^- \quad n^+ \times n^- & n^d \times n^+ \\ I^+ \mathbf{P}_{k-1}^+ & I^+ \\ n^m \times n^+ \quad n^+ \times n^- & n^m \times n^+ \end{bmatrix} \end{array}, \quad \mathscr{B}_k \equiv \mathscr{B} U_k = \begin{array}{c} \\ n^d \\ n^m \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} -\tilde{\mathbf{A}}^{0-} - \tilde{\mathbf{A}}^+ \mathbf{P}_{k-1}^+ & -\tilde{\mathbf{A}}^+ \\ n^d \times n^- \quad n^d \times n^+ \quad n^+ \times n^- & n^d \times n^+ \\ I^- & \mathbf{0} \\ n^m \times n^- & n^m \times n^+ \end{bmatrix} \end{array} \tag{A16}$$

As in the main text, we can apply the QZ decomposition to the pencil defined by $\mathscr{A}_k$, $\mathscr{B}_k$, delivering unitary Q_k and Z_k $_{(n^d+n^m)\times(n^-+n^+)}$ and upper triangular S_k and T_k $_{(n^d+n^m)\times(n^-+n^+)}$ with $Q_k^* \mathscr{B}_k Z_k = S_k$ and $Q_k^* \mathscr{A}_k Z_k = T_k$

$$\begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} T_{k,11} & T_{k,12} \\ n^- \times n^- & n^- \times n^+ \\ \mathbf{0} & T_{k,22} \\ n^+ \times n^- & n^+ \times n^+ \end{bmatrix} \end{array} \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} w_k^s \\ n^- \times n^- \\ w_k^u \\ n^+ \times n^- \end{bmatrix} = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} S_{k,11} & S_{k,12} \\ n^- \times n^- & n^- \times n^+ \\ \mathbf{0} & S_{k,22} \\ n^+ \times n^- & n^+ \times n^+ \end{bmatrix} \end{array} \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} w_k^s \\ n^- \times n^- \\ w_k^u \\ n^+ \times n^- \end{bmatrix} \mathbf{P}^- \tag{A17}$$

where

$$Z_k^* \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} I_{n^- \times n^-} \\ \Delta \mathbf{P}_k^+ \\ n^+ \times n^- \end{bmatrix} = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} w_k^s \\ n^- \times n^- \\ w_k^u \\ n^+ \times n^- \end{bmatrix} \Leftrightarrow \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} I_{n^- \times n^-} \\ \Delta \mathbf{P}_k^+ \\ n^+ \times n^- \end{bmatrix} = Z_k \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{bmatrix} w_k^s \\ n^- \times n^- \\ w_k^u \\ n^+ \times n^- \end{bmatrix} \tag{A18}$$

and

$$Z_k^* = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} Z_{k,11}^* & Z_{k,12}^* \\ n^- \times n^- & n^- \times n^+ \\ Z_{k,21}^* & Z_{k,22}^* \\ n^+ \times n^- & n^+ \times n^+ \end{bmatrix} \end{array}, \qquad Z_k = \begin{array}{c} \\ n^- \\ n^+ \end{array} \begin{array}{c} \begin{matrix} n^- & n^+ \end{matrix} \\ \begin{bmatrix} Z_{k,11} & Z_{k,12} \\ n^- \times n^- & n^- \times n^+ \\ Z_{k,21} & Z_{k,22} \\ n^+ \times n^- & n^+ \times n^+ \end{bmatrix} \end{array} \tag{A19}$$

We assume the decomposition is ordered so that the unstable eigenvalues are in the lower right blocks of S_k and T_k (hence $S_{k,22}$ and $T_{k,22}$), whence the lower block equation gives

$$T_{k,22} w_k^u = S_{k,22} w_k^u \mathbf{P}^- \Rightarrow w_k^u = T_{k,22}^{-1} S_{k,22} w_k^u \mathbf{P}^- = 0 \tag{A20}$$

as the eigenvalues of $T_{k,22}^{-1}S_{k,22}$ are inside the unit circle and $\mathbf{P}^-$ is the (semi) stable (but unknown) solvent or $\mathscr{M}$ - see the main text. Using the definitions of w_k^s and w_k^u and inserting $w_k^u = 0$ gives

$$\begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} I \\ \Delta\mathbf{P}_k^+ \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} = \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^- \end{array} & \begin{array}{c} n^+ \end{array} \\ \left[\begin{array}{c|c} Z_{k,11} & Z_{k,12} \\ \hline Z_{k,21} & Z_{k,22} \end{array}\right] \end{array}_{\substack{n^- \times n^- & n^- \times n^+ \\ n^+ \times n^- & n^+ \times n^+}} \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} w_k^s \\ \hline \mathbf{0} \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} \tag{A21}$$

and

$$\begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^- \end{array} & \begin{array}{c} n^+ \end{array} \\ \left[\begin{array}{c|c} Z_{k,11}^* & Z_{k,12}^* \\ \hline Z_{k,21}^* & Z_{k,22}^* \end{array}\right] \end{array}_{\substack{n^- \times n^- & n^- \times n^+ \\ n^+ \times n^- & n^+ \times n^+}} \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} I \\ \hline \Delta\mathbf{P}_k^+ \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} = \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} w_k^s \\ \hline \mathbf{0} \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} \tag{A22}$$

the second block equations (noting that the first block of the first definition implies $w_k^s = Z_{k,11}^{-1}$)

$$0 = w_k^u = Z_{k,21}^* + Z_{k,22}^*\Delta\mathbf{P}_k^+ \Rightarrow \Delta\mathbf{P}_k^+ = -{Z_{k,22}^*}^{-1}Z_{k,21}^* = Z_{k,21}Z_{k,11}^{-1} \tag{A23}$$

Using the definition of $\Delta\mathbf{P}_k^+$, this defines an update of $\mathbf{P}_{k-1}^+$ given by

$$\mathbf{P}_k^+ = \mathbf{P}_{k-1}^+ + \Delta\mathbf{P}_k^+ \tag{A24}$$

Upon convergence of $\mathbf{P}_k^+$, $\mathbf{P}^-$ then follows from $w_k^s = Z_{k,11}^{-1}$ and $w_k^u = 0$ inserted into the pencil

$$\begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^- \end{array} & \begin{array}{c} n^+ \end{array} \\ \left[\begin{array}{c|c} T_{k,11} & T_{k,12} \\ \hline \mathbf{0} & T_{k,22} \end{array}\right] \end{array}_{\substack{n^- \times n^- & n^- \times n^+ \\ n^+ \times n^- & n^+ \times n^+}} \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} Z_{k,11}^{-1} \\ \hline \mathbf{0} \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} = \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^- \end{array} & \begin{array}{c} n^+ \end{array} \\ \left[\begin{array}{c|c} S_{k,11} & S_{k,12} \\ \hline \mathbf{0} & S_{k,22} \end{array}\right] \end{array}_{\substack{n^- \times n^- & n^- \times n^+ \\ n^+ \times n^- & n^+ \times n^+}} \begin{array}{c} \\ n^- \\ n^+ \end{array}\!\!\begin{array}{c} \begin{array}{c} n^- \end{array} \\ \left[\begin{array}{c} Z_{k,11}^{-1} \\ \hline \mathbf{0} \end{array}\right] \end{array}_{\substack{n^- \times n^- \\ n^+ \times n^-}} \mathbf{P}_{n^- \times n^-}^- \tag{A25}$$

with the first block equation being

$$T_{k,11}Z_{k,11}^{-1} = S_{k,11}Z_{k,11}^{-1}\mathbf{P}^- \tag{A26}$$

the eigenvalue assumptions ensure that $S_{k,11}$ is invertible, hence,

$$\mathbf{P}^- = Z_{k,11}S_{k,11}^{-1}T_{k,11}Z_{k,11}^{-1} \tag{A27}$$

Finally, the (generically) non zero elements of the first n^s rows of P ($\mathbf{P}_{s,\bullet}_{n^s \times n} =$ $\begin{array}{c} \\ n^s \end{array}\!\!\begin{array}{cccc} \begin{array}{c} n^s \end{array} & \begin{array}{c} n^{--} \end{array} & \begin{array}{c} n^m \end{array} & \begin{array}{c} n^{++} \end{array} \\ \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{0} \end{array}\right] \end{array}_{n^s \times n}$) solve

$$\begin{array}{c} \\ n^s \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^{--} \end{array} & \begin{array}{c} n^m \end{array} \\ \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array}_{n^s \times n} = -\left[\underset{n^s \times n^s}{\breve{\mathbf{A}}^{0s}}\right]^{-1}\left(\underset{n^s \times n^+}{\breve{\mathbf{A}}^+}\mathbf{P}_k^+\mathbf{P}^- + \underset{n^s \times n^d}{\breve{\mathbf{A}}^{0d}}\begin{array}{c} \\ n^{--} \\ n^m \\ n^{++} \end{array}\!\!\begin{array}{cc} \begin{array}{c} n^{--} \end{array} & \begin{array}{c} n^m \end{array} \\ \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array}_{\substack{n^{--} \times n^{--} & n^{--} \times n^m \\ n^m \times n^{--} & n^m \times n^m \\ n^{++} \times n^{--} & n^{++} \times n^m}} + \underset{n^s \times n^-}{\breve{\mathbf{A}}^-}\right)$$

where from above

$$\begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{bmatrix} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{bmatrix} \end{array} = \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{bmatrix} \underset{n^{--}\times n^{-}}{\mathbf{P}^{--}} \\ \\ \underset{n^{+}\times n^{-}}{\mathbf{P}^{+}} \end{bmatrix} \end{array} \equiv \begin{array}{c} \\ n^{--} \\ n^{m} \\ n^{++} \end{array}\!\!\begin{array}{c} \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{bmatrix} \underset{n^{-}\times n^{-}}{\mathbf{P}^{-}} \\ \\ \underset{n^{++}\times n^{-}}{\mathbf{P}^{++}} \end{bmatrix} \end{array}$$

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